Conservation laws for systems of extended bodies in the first post-Newtonian approximation

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The general form of the global conservation laws for N-body systems in the first post-Newtonian approximation of general relativity is considered. Our approach applies to the motion of an isolated system of N arbitrarily composed and shaped, weakly self-gravitating, rotating, deformable bodies and uses a framework recently introduced by Damour, Soffel, and Xu (DSX). We succeed in showing that seven of the first integrals of the system (total mass-energy, total dipole mass moment, and total linear momentum) can be broken up into a sum of contributions which can be entirely expressed in terms of the basic quantities entering the DSX framework: namely, the relativistic individual multipole moments of the bodies, the relativistic tidal moments experienced by each body, and the positions and orientations with respect to the global coordinate system of the local reference frames attached to each body. On the other hand, the total angular momentum of the system does not seem to be expressible in such a form due to the unavoidable presence of irreducible nonlinear gravitational effects.

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I. INTRODUCTION

Recently, Damour, Soffel, and Xu [1–4] (DSX) developed an exhaustive approach to the first post-Newtonian dynamics of a system of N extended bodies. This theory is based on the complementary use of some local coordinate systems (attached to each body) and of a global coordinate system used to describe the orbital motion of the N bodies. Detailed analyses of the laws of global translational motion [2] and local rotational motion [3] of the bodies have been given.

In this paper, we address the question of the global conservation laws of an N-body system and their link to the quantities introduced in the DSX framework. We use the word conservation laws to mean the first integrals related to the total four momentum and angular momentum of the system, and to the center-of-mass theorem. The existence, on the first post-Newtonian level, of these ten integrals is guaranteed by the general form of the field equations [5–10]. Our main problem concerns *the form* of these conservation laws. More precisely, we investigate whether they can be entirely expressed in terms of the basic quantities introduced in the DSX post-Newtonian theory: namely, the set of the relativistic individual mass and spin multipole moments of the N bodies (M_L^A, S_L^A) and the set of the gravitoelectric and gravitomagnetic tidal moments experienced by each body (G_L^A, H_L^A). (Here, as in Refs. [1–4], whose notation we follow, $A, B = 1, \dots, N$ labels the various bodies, and $L = i_1 \cdots i_l$ is a multispatial index.) The former fully characterize the structure of the post-Newtonian gravitational field generated by an extended body in its local coordinate system, while the latter characterize the tidal action of the other bodies in this local coordinate system, including inertial contributions due to its acceleration and rotation. Obviously, the positions and velocities, with respect to the global coordinate system, of the origins of the local systems attached to each body, as well as their orientations, need to be considered, and will also enter the final expressions.

Let us start by discussing the various methods used for deriving some explicit forms of the global conservation laws in general relativity. First, let us remark that the expressions based on surface integrals at infinity [5,9,10] are of no real use within the post-Newtonian context because one loses a power $1/c^2$ in reading out a conserved quantity from the asymptotic behavior of the metric. Concerning approaches where the conserved quantities are given as three-dimensional integrals, we note the formulation of Fock [6], taken up by Brumberg [11], and that of Chandrasekhar and co-workers [7,8], followed by the parametrized version of Will [12]. Neither approach constitutes a useful starting point for us. Indeed, on the one hand, Fock and Brumberg restricted themselves to the (ill-defined) case of "rigidly rotating" bodies and introduced some mass moments, which are not compatible

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with the ones that enter naturally the DSX formalism, while, on the other hand, Chandrasekhar and Esposito restricted themselves to the special case of perfect fluids and chose basic variables that do not fit well within the DSX scheme.

A more convenient starting point for our purpose is the work of Blanchet, Damour, and Iyer [13,14] (see also [15]). These authors have defined global-frame post-Newtonian mass and spin moments of an arbitrary, isolated system of bodies in the form of integrals over the compact supports of the bodies, and they explicitly checked that the lowest moments were conserved quantities. Our task here will consist in transforming their expressions, involving integrals over the global-time simultaneity surface, into a combination of terms involving integrals over N separate local-time simultaneity surfaces. Notice that nothing guarantees *a priori* that these manipulations will lead to final expressions containing only the "good" moments introduced in the DSX scheme. In fact, we shall succeed in this task for seven of the globally conserved quantities and fail for three of them (the three components of the trickier total angular momentum).

Let us note in advance that there are several limiting cases for which it is already known that some of the globally conserved quantities can be entirely expressed in terms of some dynamically relevant individual multipole moments. First, there are the cases where one truncates the multipolar series: keeping only the monopole contributions defines the "Lorentz-Droste–Einstein-Infeld-Hoffmann" (LD-EIH) model, while also keeping the intrinsic spins of the bodies defines the "pole-dipole" (PD) model. In both cases, we dispose of well-established forms of the conservation laws (see, e.g., [5,6] for the monopole case, and [16,17] for the pole-dipole case). These will provide useful checks on our general results. Another limiting situation of possible relevance is that of a test pointlike mass moving in the gravitational field of $N-1$ extended bodies. The motion of the artificial or natural solar system satellites is a typical case of this category. Reference [4] computed explicitly the form of the corresponding Lagrangian in terms of the gravitational potentials (W, W_a), which can be algorithmically constructed using the formulas given in the Appendix of [2]. However, in this case we have first integrals only when the Lagrangian (describing geodesic motion) possesses some continuous symmetries (Noether's theorem). For instance, we can consider $N=2$ with $M_2 << M_1$ (restricted two-body problem) and with a stationary and/or axially symmetric central body (see, e.g., Ref. [18] for a study of a restricted problem of this type). This type of limiting situation will not give us useful checks.

Finally, let us remark that we hope that the present work will find practical applications in the relativistic motion of binary stars or in the celestial mechanics of the solar system. Let us recall, for instance, that one simplifies the dynamical ephemeris of the solar system (at the Newtonian approximation) by eliminating the motion of the Sun via the algebraic relation expressing that the total dipole mass moment of the solar system vanishes for all time (in a suitably mass-centered frame [19]).

In Sec. II we consecutively discuss the cases of the mass-energy integral, linear momentum integral, and briefly comment on the angular-momentum integral. Section III contains a summary of our results. Throughout the paper we follow the terminology and notation used in Refs. [1–4].

II. FIRST INTEGRALS OF THE DYNAMICAL LAWS

A. Comments on the change of time-simultaneity integration domains

As mentioned in the preceding section, the transformation of integrals from the global to several local time simultaneity surfaces is a common point to all particular cases of conservation laws. We shall thus start our discussion with a brief technical comment concerning such transformations.

Considering a specific body A, we study some integration

$$\mathcal{I}_A(t) = \int_A d^3x\, f(t,\mathbf{x}) \;, \tag{2.1}$$

performed on the global time $t=$ const surface, where $f(x^\mu)$ is some given function defined on the *compact* support of body A. We seek a transformation of the right-hand side of $\mathcal{I}_A$ such that it can be written as an integral on a local-time $T_A = T_A^0(t) =$ const simultaneity surface, say

$$\mathcal{I}_A(T_A^0(t)) = \int_A d^3X_A\, \tilde{F}(T_A^0(t), \mathbf{X}_A) \;. \tag{2.2}$$

Here, $T_A^0(t)$ denotes the value of the local time (in the reference system attached to body A) corresponding to the event on the *central* world line of the body-A reference system ($\mathbf{X}_A = 0$, also quoted as $\mathcal{L}_A$), whose global time coordinate is equal to t. In equations, if we write the spacetime coordinate transformation between the global coordinate system $x^\mu = (ct, x^i)$ and the body-A local one $X_A^\alpha = (cT_A, X_A^a)$ as

$$x^\mu = x^\mu(X_A^\alpha) = z_A^\mu(T_A) + e_{Aa}^\mu(T_A)\big[X_A^a + \frac{1}{2c^2} A_A^a \mathbf{X}_A^2 - \frac{1}{c^2}(\mathbf{A}_A\cdot\mathbf{X}_A)X_A^a\big] \;, \tag{2.3}$$

$T_A^0(t)$ is defined as the unique solution of $ct = x^0(T_A, \mathbf{0})$, i.e., $ct \equiv z_A^0(T_A^0(t))$.

First, let us denote by $f_A(T_A, \mathbf{X}_A)$ the original function $f(t,\mathbf{x})$ reexpressed in terms of the local spacetime variables X_A^α: $f_A(X_A^\alpha) \equiv f(x^\mu(X_A^\alpha))$. By mathematically transforming the variables of integration in Eq. (2.1) we get

$$\mathcal{I}_A(t) = \int_A d^3X_A \left(\left|\frac{\partial X}{\partial x}\right|^{(3)}_{t=\mathrm{const}} \right)^{-1} f_A\left[T_A\left(t,\mathbf{X}_A\right),\mathbf{X}_A\right] \;, \tag{2.4}$$

where $|\partial X/\partial x|^{(3)}_{t=\mathrm{const}}$ is the spatial Jacobian

$\det(\partial X^a/\partial x^i)$ $(a, i = 1, 2, 3)$ computed when keeping the value of t fixed and where $T_A(t, \mathbf{X}_A)$ denotes the solution of $ct = x^0(T_A, \mathbf{X}_A)$. From Eq. (2.3) (or Eqs. (A5) of Ref. [2]), the latter quantity reads explicitly

$$T_A(t, \mathbf{X}_A) = T_A^0(t) - \frac{1}{c^2} V_A^a X_A^a + O(4) ,$$

so that by expanding $f_A[T_A(t, \mathbf{X}_A), \mathbf{X}_A]$ in powers of the small time shift $(\mathbf{V}_A \cdot \mathbf{X}_A)/c^2 = O(2)$ we can express it within a sufficient accuracy in terms of functions computed on a *local-time* simultaneity surface, namely, $T_A = \text{const} = T_A^0(t)$:

$$f_A\left[T_A\left(t, \mathbf{X}_A\right), \mathbf{X}_A\right] = f_A(T_A^0(t), \mathbf{X}_A) - \frac{1}{c^2} V_A^a X_A^a \partial_T f_A\left(T_A^0(t), \mathbf{X}_A\right) + O(4) . \quad (2.5)$$

As for the three-dimensional Jacobian entering Eq. (2.4) it is easy to see that it can be expressed as

$$\left|\frac{\partial X}{\partial x}\right|^{(3)} = \left(\frac{\partial t}{\partial T_A}\right)_{X^a=\text{const}} \left|\frac{\partial X}{\partial x}\right|^{(4)} = \left[e_{A0}^0(T_A^0) + \frac{1}{c^2} A_A^a X^a\right] \left|\frac{\partial X}{\partial x}\right|^{(4)} + O(4) , \quad (2.6)$$

where $|\partial X/\partial x|^{(4)} = \det(\partial X^\alpha/\partial x^\mu)$ is the full four-dimensional Jacobian associated with the coordinate transformation (2.3). The time derivative $(\partial t/\partial T_A)$ in Eq. (2.6) is obtained from Eq. (2.3) or from putting $V_S = 0$ in expression (A6) in Appendix A of [4].

Finally, we get

$$\mathcal{I}_A(t) = \int_A d^3 X_A \left[\left|\frac{\partial X}{\partial x}\right|^{(4)}\right]^{-1} \left[e_{A0}^0\left(T_A^0\right) + \frac{1}{c^2} A_A^a X_A^a\right]^{-1} \times \left[f_A\left(T_A^0, \mathbf{X}_A\right) - \frac{1}{c^2} V_A^a X_A^a \partial_T f_A\left(T_A^0, \mathbf{X}_A\right)\right] , \quad (2.7)$$

where it is convenient to leave unexplicated the four-dimensional Jacobian because it will be directly cancelled when using the transformation laws of the mass and mass current densities σ^μ entering $f_A(t, \mathbf{x})$. For completeness, let us, however, mention its value

$$\left|\frac{\partial X}{\partial x}\right|^{(4)} = 1 - \frac{2}{c^2} W''(T, X) + O(4) , \quad (2.8)$$

where $W''(T, X) = G_A'' - A_A^a X^a + O(2)$ is the inertial contribution to the local potential due to the change of the time scale and the acceleration of the body-A frame. Here, $G_A'' = c^2 \ln(dT/d\tau_f)_A = v_A^2/2 - c^2 \ln e_{A0}^0 + O(2)$ measures the relative scaling, along the central world line $X_A^a = 0$, between the local time T_A and the global Minkowskian proper time $d\tau_f = \sqrt{-f_{\mu\nu} dz^\mu dz^\nu}/c$ (see Sec. VI E of [1]).

Note that, in geometrical terms, the mathematical transformations we have just performed correspond to using a mapping between the t = const and T_A = const $= T_A^0(t)$ hypersurfaces by means of the congruence of world lines $\mathcal{L}_{X_A^a}$ of constant spatial local coordinates (see Sec. III D in [1]).

B. Mass-energy integral

The Blanchet-Damour post-Newtonian total mass-energy $m(t)$ of an isolated system can be written as [13]

$$m(t) = \int d^3 x\, \sigma(x) - \frac{1}{c^2} \frac{d}{dt} \int d^3 x \left(\sigma^i x^i\right) + O(4) , \quad (2.9)$$

where σ_μ are densities of mass and mass currents [$\sigma = (T^{00} + T^{ii})/c^2$ and $\sigma^i = T^{0i}/c$; $T^{\mu\nu}$ denoting the components of the stress-energy tensor in the *global* coordinate system]. The integration in (2.9) is to be performed over a global-time t = const hypersurface spanning the whole N-body system. However, as σ^μ is nonzero only in the neighborhood of the bodies, we can directly use the results of the previous subsection to decompose $m(t)$ as a sum of N terms integrated over local simultaneity surfaces $T_A^0(t)$ = const.

Let us recall the transformation law between the global and local coordinate systems of the mass densities pertaining to a given point of body A:

$$\sigma(x) = \left|\frac{\partial X}{\partial x}\right|^{(4)} \left[\left(1 + 2\frac{v_A^2}{c^2}\right) \Sigma(X) + \frac{4}{c^2} V_A^a \Sigma_A^a(X)\right] + O(4) , \quad (2.10)$$

as given in [1].

Putting together the definition (2.9) and the method explained in Sec. II A [where we note that the four-dimensional Jacobian cancels between Eqs. (2.7) and (2.10)] we arrive after some algebra at

$$m(t) = \sum_A \left[M^A \left(1 + \frac{1}{2c^2} v_A^2\right) + \frac{1}{c^2} \frac{d}{dT_A} \left(M_a^A V_A^a\right) + \frac{1}{2c^2} \sum_l \frac{2l+1}{l!} M_L^A \left(G_L^A + G_L^{A\prime\prime}\right)\right] + O(4) , \quad (2.11)$$

where all quantities on the right-hand side must be evaluated at the intersection of the t = const hypersurface with the central world line of the corresponding body [i.e., for $T_A = T_A^0(t)$]. We recall that the quantities M^A (which are not constant in general) denote the local, individual relativistic mass monopoles of each body. Each M^A is a directly observable quantity in the sense that it is just the gravitational mass measured from interpreting the locally measured orbital motion of artificial or natural satellites around body A. Similarly the M_L^A (for $l \geq 1$) are the locally measured mass multipole moments of body A, and G_L^A the locally felt tidal moments. The other quantities entering Eq. (2.11) are related to the way the local A-reference system is moving with respect to the global coordinate system [in particular $G''^{A}_{L} = (G''^{A}, -A^{A}_{a}, +3A^{A}_{<a}A^{A}_{b>}/c^2, 0, 0, ...)$ measure the inertial contributions to the tidal moments felt in the local A system]. The result (2.11) is new, and its precise form (e.g., the numerical factor $2l+1$ in front of the $M^A_L G^A_L$ product) is different from what one might have naively expected from the standard Newtonian expression for the total energy (e.g., [20]).

As discussed in detail in Ref. [1] the DSX framework leaves open some freedom in fixing several quantities related to the origin an orientation of the local coordinate systems, as well as the gauge for the time coordinate along the world line $\mathcal{L}_A$. We shall call "standard world line data" the case where this freedom is used to satisfy the following constraints: (i) $\forall A, \forall T_A, M^A_a(T_A) = 0$ (which means identifying the origin of all local coordinate systems with the relativistic mass centers of the bodies), (ii) $\forall A, \forall T_A, \bar{W}^A_\alpha(T_A, 0) = 0$ (the so-called weak effacement condition for the external gravitational potential in the local frames). Note that we still leave unconstrained the orientation of the local frame axes, which can undergo a slow rotation described by the matrix $R^i_{Aa}(T)$. In the case of the standard world line data, Eq. (2.11) simplifies to the form

$$m^{\text{standard}}(t) = \sum_A \left\{ M^A \left[1 + \frac{1}{2c^2}\left(v_A^2 - G'_A\right)\right] + \frac{1}{2c^2}\sum_{l\geq 2}\frac{2l+1}{l!}M^A_L G^A_L \right\} + O(4) , \quad (2.12)$$

where $G'_A = \sum_{B\neq A} G^{B/A} = \sum_{B\neq A} w^B(z_A) + O(2)$ [$G'_A = -G''_A$ for standard data] denote the value on the central world line $\mathcal{L}_A$ of the Newtonian potential generated by all the other bodies [$w^B(z_A) = \sum_{l\geq 0}\frac{(-)^l}{l!}GM^B_L\partial^A_L|z_A - z_B|^{-1}$].

In the monopole (LD-EIH) or pole-dipole limit Eq. (2.12) yields the well-known result that the total mass-energy is the sum of the total rest-mass and the kinetic and potential ($\propto GM^AM^B/|z_A - z_B|$) energy terms. Let us recall that, in the general case, the individual post-Newtonian gravitational masses M^A are no longer constant (because of tidal forces acting on the extended bodies; see Eqs. (4.20a) and (4.21a) of [2]) and that it is not *a priori* evident that the quantities $m(t)$ or $m^{\text{standard}}(t)$ are conserved. As a check on our algebra, we have verified by a direct calculation that this is indeed the case.

As an aside, let us conclude this subsection by noting that if one defines, as an auxiliary technical quantity, the "Fock mass" of the Ath body by the relation $c^2 M^A_F = \int_A d^3X_A\,(1 + W/2c^2)\mathbf{T}^{00}_A$ (where $\mathbf{T}^{00}_A$ denote the X^0_A-X^0_A component of the stress-energy tensor considered in the local coordinate system X^α_A), one can write our result (2.12) in a formally compact (and familiar looking) form. The name we give to this quantity is based on the fact that in his book [6] Fock used such an expression for the total mass-energy. Note, however, that he always used it in the global coordinate system. Our definition is written in the local system X^α_A, but we use the *total* potential in the local frame for $W(T, X)$, containing both internally and externally (including inertially) generated contributions. By using the expressions (4.15) of [2] for the tidal expansion of $W(T, X)$ we find (in any world line gauge) the following relation between the Blanchet-Damour mass and the Fock one:

$$M^A = M^A_F - \frac{1}{2c^2}\sum_l \frac{2l+1}{l!}M^A_L G^A_L + O(4) . \quad (2.13)$$

Thus in the case of standard data we can rewrite our previous formula (2.12) in the following form:

$$m^{\text{standard}}(t) = \sum_A M^A_F \left[1 + \frac{1}{2c^2}\left(v_A^2 - G'_A\right)\right] + O(4) . \quad (2.14)$$

Let us, however, emphasize again that it is only the Blanchet-Damour mass M^A that is directly observable as a gravitational mass determining the orbital motion of satellites of body A. The Fock mass M^A_F is just a mathematical construct.

C. Center-of-mass integral and linear momentum

The Blanchet-Damour post-Newtonian dipole mass moment $m_i(t)$ of the whole system [13] can be written as

$$m_i(t) = \int d^3x\, x^i \sigma(x) - \frac{1}{c^2}\frac{d}{dt}\int d^3x\, \sigma^j\left(x^i x^j - \frac{1}{2}\delta_{ij}x^2\right) + O(4) . \quad (2.15)$$

It satisfies the conservation law [13]

$$\frac{d^2 m_i(t)}{dt^2} = 0 . \quad (2.16)$$

In fact, the first derivative of $m_i(t)$ is nothing but the conserved total linear momentum of the system:

$$p_i \equiv \frac{dm_i(t)}{dt} = \text{const} . \quad (2.17)$$

Employing the results of Sec. II A and definition (2.15) we obtain, after tedious but straightforward calculations, the following form of the total mass dipole moment $m_i(t)$ of the system:

$$\begin{aligned} m_i(t) = \sum_A \Bigg\{ & M^A z^i_A \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v_A^2 + G''_A\right)\right] + \frac{1}{c^2}v^j_A s^{ij}_A - \frac{3}{c^2}a^j_A m^{ij}_A \\ & - \frac{1}{c^2}\left(z^i_A z^j_A - \frac{1}{2}\delta_{ij}z^2_A\right) R^j_{Aa}\sum_l \frac{1}{l!}M^A_L G^{A\prime}_{aL} - \frac{2}{c^2}z^j_A R^i_{A[a}R^j_{Ab]}\sum_l \frac{1}{l!}M^A_{aL}G^{A\prime}_{bL} \Bigg\} \\ & + \sum_A \left\{ m^i_A \left[1 + \frac{1}{c^2}\left(\frac{1}{2}v_A^2 + G''_A\right)\right] + \frac{1}{c^2}z^i_A\left[M^{A(1)}_a V^a_A + 2M^A_a G^{A\prime\prime}_a\right]\right\} + O(4) , \end{aligned} \quad (2.18)$$

where $s_A^{ij} = \epsilon_{ijk} s_A^k = \epsilon_{ijk} R^k_{Aa} S^a_A$, $m_A^{i_1 i_2 \cdots i_n} = e^{i_1}_{Aa_1} e^{i_2}_{Aa_2} \cdots e^{i_n}_{Aa_n} M_A^{a_1 a_2 \cdots a_n}$, where $M_a^{A(1)} = dM_a^A / dT_A$ and where the square brackets in $R^i_{A[a} R^j_{Ab]}$ mean antisymmetrization $[u_{[a} v_{b]} \equiv \frac{1}{2}(u_a v_b - u_b v_a)]$. It should be emphasized that the fact that the final expression (2.18) can be entirely written in terms of the good moments entering the DSX framework is far from being a trivial result. In the intermediate calculations the "bad" moments (N_L, P_L) defined in Eqs. (4.22) of [2] enter at several places before finally cancelling. We also remark that the last sum in curly brackets vanishes if one uses standard world line data, as defined above.

As mentioned in Sec. I, we can get partial checks on our results by considering models where the multipole series is highly truncated. In particular, if we keep only the mass monopoles of the bodies (LD-EIH limit), formula (2.18) reduces to

$$m^i_{\text{LD-EIH}} = \sum_A M^A z^i_A \left[1 + \frac{1}{2c^2}\left(v_A^2 - G'_A\right)\right] + O(4) = \sum_A M^A z^i_A \left[1 + \frac{1}{2c^2}\left(v_A^2 - \sum_{B \neq A} \frac{GM^B}{r_{AB}}\right)\right] + O(4) \;, \qquad (2.19)$$

where the second row applies to the case of the standard world line data. Equation (2.19) agrees with previous results [5,10]. In the case of the pole-dipole truncated model the expression for the mass dipole reads

$$m^i_{\text{PD}} = m^i_{\text{LD-EIH}} + \frac{1}{c^2} \sum_A v^j_A s^{ij}_A + O(4) \;, \qquad (2.20)$$

a result previously derived by Damour and Schäfer [17] from the spin-dependent Lagrangian of Ref. [16].

Let us remark, as an aside, that defining some "Fock" local mass dipole moments for instance by $c^2 M^A_{aF} = \int_A d^3X_A\, X^a_A (1 + W/2c^2) \mathbf{T}^{00}$ does not at all simplify the writing of our result (2.18). In fact, this definition introduces several bad algebraic structures (notably the moments N^A_L; see [1–3]), which do not enter the final dynamical results of the DSX formalism. This is one of the reasons why Fock, in his book [6], did not succeed in getting a good definition of the mass centers of the individual bodies (in spite of the fact that, in the case of the mass center of the entire isolated system, the Blanchet-Damour and Fock definitions, written in the global coordinate system, give the same result; see Eq. (3.45) of [13]).

From the result (2.17) above, we can easily derive the following explicitly DSX-like expression for the total linear momentum of the N-body system:

$$\begin{aligned} p_i(t) = \sum_A \Big\{ & M^A v^i_A \left[1 + \frac{1}{c^2}\left(\frac{1}{2} v_A^2 + G''_A\right)\right] + \frac{1}{c^2} a^j_A s^{ij}_A - \frac{3}{c^2}\frac{d}{dt}\left(a^j_A m^{ij}_A\right) \\ & - \frac{1}{c^2} z^i_A \sum_l \frac{1}{l!}\left[l M^{A(1)}_L G^{A\prime}_L + (l+1)\, M^A_L G^{A(1)\prime}_L\right] - \frac{1}{c^2}\left(v^i_A z^j_A - \delta_{ij} \mathbf{v}_A . \mathbf{z}_A\right) R^j_{Aa} \sum_l \frac{1}{l!} M^A_L G^{A\prime}_{aL} \\ & - \frac{1}{c^2}\left(z^i_A z^j_A - \frac{1}{2}\delta_{ij} z^2_A\right) R^j_{Aa} \sum_l \frac{1}{l!}\frac{d}{dT_A}\left(M^A_L G^{A\prime}_{aL}\right) - \frac{2}{c^2} z^j_A R^i_{A[a} R^j_{Ab]} \sum_l \frac{1}{l!}\frac{d}{dT_A}\left(M^A_{aL} G^{A\prime}_{bL}\right) \Big\} \\ & + \sum_A \left\{ \frac{d}{dT_A}\left(m^i_A\right)\left(1 + \frac{2}{c^2} G''_A\right) + \frac{1}{c^2} v^j_A \frac{d}{dt}\left(m^j_A v^i_A\right) + \frac{1}{c^2} m^i_A \frac{d}{dT_A} G''_A - \frac{2}{c^2} v^i_A m^j_A a^j_A \right\} + O(4) \;. \end{aligned} \qquad (2.21)$$

Again, the last sum of terms in curly brackets disappears in the standard world line gauge.

The LD-EIH form of the linear momentum is obtained by retaining only the mass monopole terms

$$p^i_{\text{LD-EIH}} = \sum_A \left\{ M^A v^i_A \left[1 + \frac{1}{2c^2}\left(v_A^2 - \sum_{B \neq A} \frac{GM^B}{r_{AB}}\right)\right] - \frac{G}{2c^2} \sum_{B \neq A} \frac{GM^A M^B}{r_{AB}} \left(\mathbf{n}_{AB} \cdot \mathbf{v}_B\right) n^i_{AB} \right\} + O(4) \;. \qquad (2.22)$$

The presence of spin dipoles (in the framework of the PD model) results in the additional term

$$p^i_{\text{PD}} = p^i_{\text{LD-EIH}} + \frac{1}{c^2} \sum_A a^j_A s^{ij}_A + O(4) \;, \qquad (2.23)$$

as mentioned in [17].

As in the case of the total mass-energy, but with more work, one can directly check that the global-time derivative of Eq. (2.21) vanishes to the indicated accuracy. Because of the incompatibilities between the DSX and the Fock approaches mentioned above, it is not possible to compare directly our results with those given by Fock [6] or Brumberg [11].

D. Angular momentum

Damour and Iyer [14] and Ref. [3] have shown that the treatment of the local individual spin of bodies, members of an interacting N-body system, faces serious problems because of the unavoidable intervention of nonlinear, interbody gravitational effects. Mathematically, this complication manifests itself through the occurrence of bad DSX moments (N_L, P_L). It has been, however, possible to reach a successful formulation of the individual rotational laws of motion through a carefully adjusted definition of the individual spin of each body [3]. In the following, we briefly address the problem of breaking up the total angular momentum into a sum of contributions which make sense within the DSX framework.

The global angular momentum of the system reads [6,14,3]

$$s_i(t) = \epsilon_{ijk} \int d^3x\, x^j \left\{ \sigma^k \left[1 + \frac{4}{c^2} w \right] - \frac{\sigma}{c^2} \left[4w^k + \frac{1}{2} \partial_k \partial_t z(x,t) \right] \right\} + O(4) \,, \quad (2.24)$$

where $z(x,t) = G \int d^3x'\, \sigma(x',t) |\mathbf{x} - \mathbf{x}'|$. For an isolated system of arbitrary bodies it has been previously shown that $s_i(t)$ is conserved at the post-Newtonian level.

In order to preserve the post-Newtonian accuracy [i.e., modulo $O(4)$] of the result, one needs the transformation law of the global mass current $\sigma^k(x)$ to local coordinate quantities with corresponding precision. After some algebra one arrives at

$$\begin{aligned} \sigma^i(x) = \left| \frac{\partial X}{\partial x} \right|^{(3)} \Bigg\{ & v^i_A e^0_{A0} \Sigma + e^i_{Aa} \Sigma^a + \frac{1}{c^2} \left[2 v^i_A G''_A + c^2 e^{i(1)}_{Aa} X^a - v^i_A A_{Aa} X^a + R^i_{Aa} \left(\frac{1}{2} A^{(1)}_{Aa} X^2 - A^{(1)}_{Ab} X^b X^a \right) \right] \Sigma \\ & + \frac{1}{c^2} \left[2 R^i_{Aa} G''_A + v^i_A V^a_A - R^i_{Aa} A^b_A X^b + R^i_{Ab} \partial_a \left(\frac{1}{2} A^b_A X^2 - A^c_A X^c X^b \right) \right] \Sigma^a \\ & - \frac{1}{c^2} \left[v^i_A \partial_T \left(X^b V^b_A \Sigma \right) + R^i_{Aa} \partial_T \left(X^b V^b_A \Sigma^a \right) \right] + \frac{1}{c^2} V^a_A R^i_{Ab} \left[\mathbf{T}^{ab} - \delta_{ab} \mathbf{T}^{cc} \right] \Bigg\} + O(4) \end{aligned} \quad (2.25)$$

(where $e^{i(1)}_{Aa} = de^i_{Aa}/dT_A$, etc.), generalizing the formula $\sigma^i = v^i_A \Sigma + R^i_{Aa} \Sigma^a + O(2)$ used throughout the series of papers [1–4]. [Note the three-dimensional Jacobian, as defined in Eq. (2.6) above, in front of Eq. (2.25).]

Inserting this relation into the defining integral (2.24) and using the method of Sec. II A one obtains an expression of the form

$$s_i(t) = \sum_A \left[\Psi^A_i \left(M_L, S_L, G_L, H_L; P_L, N_L \right) + \theta^A_{1i} + \theta^A_{2i} \right] \,, \quad (2.26)$$

where Ψ^A_i is a function of both the good DSX moments (M_L, S_L, G_L, H_L) and the bad ones (P_L, N_L) and where the remaining two terms [which follow from the last term in Eq. (2.25)] read

$$\begin{aligned} c^2 \theta^A_{1i} &= \epsilon_{ijk} z^j_A V^a_A R^k_{Ab} \textstyle\int_A d^3X_A \left[\mathbf{T}^{ab} - \delta_{ab} \mathbf{T}^{cc} \right] \,, \\ c^2 \theta^A_{2i} &= \epsilon_{ijk} R^j_{Aa} V^b_A R^k_{Ac} \textstyle\int_A d^3X_A\, X^a \left[\mathbf{T}^{bc} - \delta_{bc} \mathbf{T}^{dd} \right] \,. \end{aligned}$$

Note that if we augment our list of bad moments by including $Q^A_{ab} = \int_A d^3X_A\, \mathbf{T}^{ab}$ and $Q^A_{ab;c} = \int_A d^3X_A\, \mathbf{T}^{ab} X^c_A$, we can express the total spin $s_i(t)$ in terms of some individual moments of the N bodies. We tried to get rid of the nondynamical moments (P^A_L, N^A_L, Q^A_L) by using the local conservation of energy-momentum $(\nabla_\alpha \mathbf{T}^{\alpha\beta} = 0)$ to connect them to the dynamical moments (M^A_L, S^A_L) and the tidal ones (G^A_L, H^A_L). We did not succeed in doing so. [In fact, such transformations introduced an undesirable dependence upon the internal part $W^{\alpha +}_A(X)$ of the local gravitational potential.] We thus hypothesize that the total spin s_i of the system is not algebraically reducible to the DSX dynamical quantities.

III. CONCLUSION

The algebraic form of the global relativistic conservation laws has been examined within the perspective of the DSX post-Newtonian dynamics of an N-body system. We succeeded in breaking up seven of these conservation laws (mass-energy, center-of-mass quantity, and linear momentum) into a sum of individual contributions involving only the basic dynamical quantities of the DSX formalism. The angular momentum conservation law resisted, however, our efforts.

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