N-Body Lectures

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Introduction to NBODY6

Hermite Integration

Neighbour Scheme

Two-Body Regularization

Compact Subsystems

Post-Newtonian Treatment

Newton's Equations

Force $\mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$

Explicit differentiation

$$\mathbf{F}_i^{(1)} = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} - 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

New solution at $t = \Delta t$

$$\Delta \dot{\mathbf{r}}_i = (\frac{1}{2}\mathbf{F}_i^{(1)}\Delta t + \mathbf{F}_i)\Delta t$$

$$\Delta \mathbf{r}_i = ((\frac{1}{6}\mathbf{F}_i^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_i)\Delta t + \dot{\mathbf{r}}_i)\Delta t$$

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\begin{aligned}
\mathbf{F} &= \mathbf{F}_0 + \mathbf{F}_0^{(1)}\, t + \frac{1}{2}\mathbf{F}_0^{(2)}\, t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\, t^3 \\
\mathbf{F}^{(1)} &= \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\, t + \frac{1}{2}\mathbf{F}_0^{(3)}\, t^2
\end{aligned}$$

Prediction

$$\begin{aligned}
\mathbf{r}_j &= ((\frac{1}{6}\mathbf{F}_0^{(1)}\,\delta t_j' + \frac{1}{2}\mathbf{F}_0)\,\delta t_j' + \mathbf{v}_0)\,\delta t_j' + \mathbf{r}_0 \\
\mathbf{v}_j &= ((\frac{1}{2}\mathbf{F}_\mathbf{0}^{(1)}\,\delta t_j' + \mathbf{F}_0)\,\delta t_j' + \mathbf{v}_0\,; \qquad \delta t_j' = t - t_0
\end{aligned}$$

New forces $\qquad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\begin{aligned}
\mathbf{F}_0^{(3)} &= (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{6}{t^3} \\
\mathbf{F}_0^{(2)} &= (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\, t)\, \frac{2}{t^2}
\end{aligned}$$

Corrector for i

$$\begin{aligned}
\Delta\mathbf{r}_i &= \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5 \\
\Delta\mathbf{v}_i &= \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4
\end{aligned}$$

Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0)$$

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales

$$\Delta t_n << \Delta t_d, \quad n << N$$

Neighbour sphere $R_s^{\text{new}} = R_s^{\text{old}} \left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s, \quad \Rightarrow \ list$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Basic Regularization

Two-body equation $\ddot{x} = -\frac{M}{x^2}$

Smoothing function $t' \equiv \frac{dt}{d\tau} = x$

Rule of differentiation $\frac{d}{dt} = \frac{1}{x}\frac{d}{d\tau}$

Time-smoothed equation $x'' = \frac{x'^2}{x} - M$

Binding energy $h = \frac{1}{2}\dot{x}^2 - \frac{M}{x}$

Substitution $\dot{x} = \frac{x'}{x} \Rightarrow$ $x'' = 2hx + M$

Coordinate transformation $u^2 = x$

Twice diff. of u^2 and $h \Rightarrow$ $u'' = \frac{1}{2}hu$

Regular equation for $x \Rightarrow 0$

Levi-Civita Formulation

2D system: u_1, u_2

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 \\ R_2 &= 2u_1u_2 \end{aligned}$$

Transformation

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u}$$

Levi-Civita [1920] matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \qquad \Rightarrow \ R = u_1^2 + u_2^2$$

Definition $\dot{\mathbf{R}}^2 = \dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}}$ with $\mathbf{R}' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'$ and $\dot{R} = R'/R$

$$\dot{\mathbf{R}} = 2\mathcal{L}(\mathbf{u})\mathbf{u}'/R$$

$\dot{\mathbf{R}}^T = 2\mathbf{u}'\mathcal{L}^T(\mathbf{u})/R$ and $\mathcal{L}^T(\mathbf{u})\mathcal{L}(\mathbf{u}) = R\mathbf{I}$ give

$$\dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}} = 4\mathbf{u}' \cdot \mathbf{u}'/R$$

From $\mathcal{L}'(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$ we have $\mathbf{R}'' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'' + 2\mathcal{L}(\mathbf{u}')\mathbf{u}'$

Final equation of motion, with $\mathbf{u} \cdot \mathbf{u} = R$

$$\mathbf{u}'' = \frac{1}{2}h\mathbf{u} + \frac{1}{2}R\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Binding energy per unit reduced mass

$$h = [(2\mathbf{u}' \cdot \mathbf{u}' - (m_k + m_l)]/R$$

Rate of change from $\dot{\mathbf{R}} \cdot \ddot{\mathbf{R}}$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{R}}^2 - \frac{(m_k + m_l)}{R}\right] = \dot{\mathbf{R}} \cdot \mathbf{F}_{kl}$$

Conversion by $h' = \mathbf{R}' \cdot \mathbf{F}_{kl}$ and $\dot{\mathbf{R}}$

$$h' = 2\mathbf{u}' \cdot \mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

KS Regularization

New coordinates $R \; = \; u_1^2 + u_2^2 + u_3^2 + u_4^2$

Time transformation $dt \; = \; R \, d\tau$

Coordinate transformation $\mathbf{R} \; = \; \mathcal{L}(\mathbf{u}) \, \mathbf{u}$

Levi-Civita matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{bmatrix}$$

Equations of motion

$$\begin{aligned} \mathbf{u}'' &= \frac{1}{2} h \, \mathbf{u} + \frac{1}{2} R \, \mathcal{L}^{\mathcal{T}} \, \mathbf{P} \\ h' &= 2 \, \mathbf{u}' \cdot \mathcal{L}^{\mathcal{T}} \, \mathbf{P} \\ t' &= \mathbf{u} \cdot \mathbf{u} \end{aligned}$$

Close encounter $\Delta t_i \; < \; \Delta t_{cl} \, ; \quad R \; < \; r_{cl}$

Termination $\gamma \; \equiv \; \dfrac{|\mathbf{P}| \, R^2}{m_i \, + \, m_j} \; > \; 0.5$

Centre of mass motion $\ddot{\mathbf{r}} \; = \; \dfrac{m_i \, \mathbf{P}_i \; + \; m_j \, \mathbf{P}_j}{m_i \; + \; m_j}$

Perturber selection $r_k \; < \; \lambda \, R \, , \quad \gamma \; > \; 1 \times 10^{-6}$

Three-Body Regularization

Initial conditions $\mathbf{r}_i,\ \mathbf{p}_i, \quad \mathbf{p}_3 = -(\mathbf{p}_1 + \mathbf{p}_2)$

Basic Hamiltonian $\mu_{k3} = \dfrac{m_k m_3}{m_k + m_3}$

$$H = \sum_{k=1}^{2} \frac{1}{2\mu_{k3}} \mathbf{p}_k^2 + \frac{1}{m_3} \mathbf{p}_1 \cdot \mathbf{p}_2 - \frac{m_1 m_3}{R_1} - \frac{m_2 m_3}{R_2} - \frac{m_1 m_2}{R}$$

KS coordinate transformation $\mathbf{Q}_k^2 = R_k\,, \quad (k = 1, 2)$

Time transformation $dt = R_1 R_2\, d\tau$

Regularized Hamiltonian $\Gamma^* = R_1 R_2\, (H - E_0)$

$$\Gamma^* = \sum_{k=1}^{2} \frac{1}{8\mu_{k3}} R_l\, \mathbf{P}_k^2 + \frac{1}{16 m_3} \mathbf{P}_1^T \mathbf{A}_1 \cdot \mathbf{A}_2 \mathbf{P}_2$$

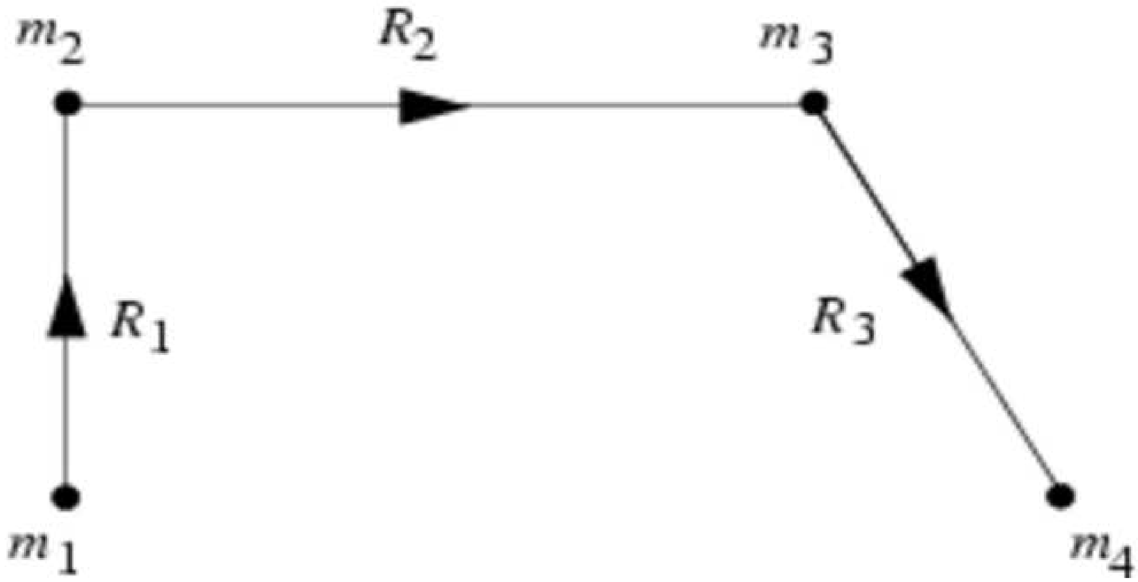
$$- m_1 m_3 R_2 - m_2 m_3 R_1 - \frac{m_1 m_2 R_1 R_2}{|\mathbf{R}_1 - \mathbf{R}_2|} - E_0 R_1 R_2$$

Equations of motion

$$\frac{d\mathbf{Q}_k}{d\tau} = \frac{\partial \Gamma^*}{\partial \mathbf{P}_k}\,, \qquad \frac{d\mathbf{P}_k}{d\tau} = -\frac{\partial \Gamma^*}{\partial \mathbf{Q}_k}$$

Regular solutions: $R_1 \to 0$ or $R_2 \to 0$

Singular term < regular terms: $|\mathbf{R}_1 - \mathbf{R}_2| > \max\,(R_1,\ R_2)$


m_2
R_2
m_3
R_1
R_3
m_1
m_4

Chain Strategy

Initialization	$\Delta t_{\mathrm{cm}} < \Delta t_{\mathrm{min}} \;\Rightarrow\; KS + m_p$
External perturbation	$\gamma_{\mathrm{ch}} = \dfrac{2P_x R_p^3}{M_{\mathrm{ch}}}, \quad R_p = \dfrac{1}{2}\sum R_i$
Membership check	Gain/loss; inject $\gamma_{\mathrm{ch}} > 0.1$
Strong perturber	$\gamma_{\mathrm{ch}} > 0.05,\; \dot{d} < 0,\; d < 5R_{\mathrm{cl}}$
Three-body stability	Every 1000 steps
Four-body stability	Not implemented in ARC
PN treatment	IPN = 1, 2, 3
Einstein shift	$\dfrac{6\pi M}{ac^2(1-e^2)} > 1 \times 10^{-4}$

ARC treatments

Standard termination pert $> 0.02, R_{sum} > 5R_{min}$

Injection $R_p < 5R_{min}, \dot{R}_p < 0$

Collision $a(1-e) < 2R_{coll}$

Stability condition $a_1(1-e_1) > R_{stab}$

GR coalescence $a(1-e) < 4GM/c^2$

Chain selection: X–X————————X

X–X——————-X–X

Removal: IESC, JESC from ISORT & INAME

Relativistic Effects

Einstein shift

$$\Delta\omega = \frac{6\pi M}{ac^2(1-e^2)}$$

GR radiation time-scale

$$t_{\rm GR} = \frac{5}{64}\frac{c^5\, g(e)\, a^4}{X(1+X)\, m_1^3}$$

$$X = \frac{m_2}{m_1}, \qquad g(e) \simeq \frac{(1-e^2)^{7/2}}{4.5}$$

Speed of light & Schwarzschild radius

$$c = \frac{3\times 10^5}{V^*}, \quad R_{\rm Sch} = \frac{2M}{c^2}$$

Velocities

$$V^* = \alpha\left(\frac{M}{R_{pc}}\right)^{1/2}, \quad v_{\rm max} = \left(\frac{M}{a}\right)^{1/2}\left(\frac{1+e}{1-e}\right)^{1/2}$$

PN conditions

$$\frac{v}{c} > f, \qquad t_{\rm GR} < t^*, \qquad r < n\, R_{\rm Sch}$$

Post-Newtonian Terms

Equation of motion $$\frac{d^2\mathbf{r}}{dt^2} = \frac{M}{r^2}\left[(-1+A)\frac{\mathbf{r}}{r} + B\mathbf{v}\right]$$

GR formulation: Blanchet & Iyer 2003

First-order precession $$M = m_1 + m_2, \quad \eta = \frac{m_1 m_2}{M^2}$$

$$A_1 = 2(2+\eta)\frac{M}{r} - (1+3\eta)v^2 + \frac{3}{2}\eta\dot{r}^2$$

$$B_1 = 2(2-\eta)\dot{r}$$

Higher-order precession $$A_2 = ..., \quad B_2 = ..., \quad A_3 = ..., \quad B_3 = ...$$

Gravitational radiation $$A_{5/2} = \frac{8}{5}\eta\frac{M}{r}\dot{r}\left(\frac{17M}{3r} + 3v^2\right)$$

$$B_{5/2} = -\frac{8}{5}\eta\frac{M}{r}\left(3\frac{M}{r} + v^2\right)$$

Total GR perturbation

$$\mathbf{P}_{GR} = \frac{M}{c^2r^2}\left[(A_1 + \frac{A_2}{c^2} + \frac{A_{5/2}}{c^3})\frac{\mathbf{r}}{r} + (B_1 + \frac{B_2}{c^2} + \frac{B_{5/2}}{c^3})\mathbf{v}\right]$$

Energy check $$E_{\rm tot} - \int \mathbf{P}_{\rm GR}\cdot\mathbf{v}\,dt = \text{const}$$

GR radiation time-scale $$t_{\rm GR} = \frac{5}{64}\frac{c^5\,g(e)\,a^4}{X(1+X)\,m_1^3}, \quad c = \frac{3\times10^5}{V^*}$$

$$g(e) \simeq \frac{(1-e^2)^{7/2}}{4.35}, \quad X = \frac{m_2}{m_1}$$

Units

(a) Scaling relations

Given length scale $R_{\rm V}$ in pc and total mass $NM_{\rm S}$ in $M_\odot$

Velocity scaling

$$\tilde{V}^* = 1 \times 10^{-5} (GM_\odot/L^*)^{1/2} \text{ km/s, with } L^* = 3 \times 10^{18} \text{ cm}$$

Velocity unit $\quad V^* = 6.557 \times 10^{-2} (NM_{\rm S}/R_{\rm V})^{1/2}\,\text{km/s}$

Fiducial time $\quad \tilde{T}^* = (L^{*3}/GM_\odot)^{1/2} = 14.94 \text{ Myr}$

Time unit $\quad T^* = 14.94\,(R_{\rm V}^3/NM_{\rm S})^{1/2} \text{ Myr}$

(b) Conversion from N-body to physical units

$$\tilde{r} = R_{\rm V}\, r \text{ pc}, \quad \tilde{v} = V^*\, v\ km/s, \quad \tilde{t} = T^*\, t \text{ Myr},$$
$$\tilde{m} = NM_{\rm S}\, m\ M_\odot$$

Crossing time $\quad T_{\rm cr} = 2\sqrt{2}\, T^* \text{ Myr}$

Data Structure

Single stars	$2\,N_{\rm p} < i \le N, \quad \mathcal{N}_i = i$
KS pairs	$1 \le i \le 2\,N_{\rm p}\,, \quad i_p = i_{\rm icm} - N$
C.m. particles	$i > N, \quad \mathcal{N} = N_0 + \mathcal{N}_k\,, \quad k = 2i_p - 1$
Stable triples	$\mathrm{KS} + \mathrm{ghost}, \quad \mathcal{N}_{\rm cm} = -\mathcal{N}_k$
Ghost particles	$\mathcal{N}_{\rm ghost} = \mathcal{N}_{2i_{\rm p}-1}\,, \quad m_{\rm ghost} = 0$
Stable quadruples	$\mathrm{KS} + \mathrm{KS\ ghost}, \quad \mathcal{N}_{\rm cm} = -\mathcal{N}_k$
Higher orders	$\mathrm{T} + \mathrm{KS}, \quad \mathcal{N}_{\rm cm} = -(2N_0 + \mathcal{N}_k)$
Chain members	$2\,N_{\rm p} < i_{\rm cm} \le N, \quad \mathcal{N}_{\rm cm} = 0$
Single escape	$2N_{\rm p} < i \le N, \quad r_i > 2r_{\rm tide}\,, \quad \mathrm{remove}\ i$
Binary escape	$i > N\,, \quad r_i > 2r_{\rm tide}\,, \quad 2i_p - 1,\ 2i_p$
Hierarchy escape	$i > N\,, \quad r_i > 2r_{\rm tide}\,, \quad 2i_p - 1,\ 2i_p\,,\ i_{\rm ghost}$

Scaling of Initial Conditions

Main input	$N,\ N_{\rm b},\ M_{\rm S},\ R_{\rm pc}$
Cluster parameters	optional IMF and Plummer or King model
Initial data	$\tilde{m}_i,\ \tilde{\mathbf{r}}_i,\ \tilde{\mathbf{v}}_i, ...,\ i = 1,\ N$
Total energy	$E = T - U$
Virial theorem	$\mathbf{v}_i = q\,\tilde{\mathbf{v}}_i,\quad q = \left[\dfrac{Q_{\rm V} U}{T}\right]^{1/2},\quad \mathbf{r}_i = \tilde{\mathbf{r}}_i$
Standard units	$G = 1,\ \ \sum m_i = 1,\ \ E_0 = -0.25$
Standard scaling	$\hat{\mathbf{r}}_i = \dfrac{\mathbf{r}_i}{S^{1/2}},\ \hat{\mathbf{v}}_i = \mathbf{v}_i\,S^{1/2},\ S = \dfrac{E_0}{q^2 T - U}$
Astrophysical units	$V^*,\ T^*,\ R^*$ from $M_{\rm tot},\ R_{\rm pc}$
Primordial binaries	split or copy m_i, introduce $a,\ e,\ \Omega$
Force polynomials	$\mathbf{F}_i,\ \dot{\mathbf{F}}_i,\ \Delta t_i, ...,\ i = 1,\ N$
KS regularization	explicit initialization, $R < R_{\rm cl}$

Essential Input Parameters

Particle numbers	$N,\ n_{\rm max},\ N_{\rm crit}$
Integration variables	$\eta_{\rm I},\ \eta_{\rm R},\ S_0,\ \Delta T,\ T_{\rm crit},\ Q_{\rm E},\ R_{\rm pc},\ \bar{m}$
Optional procedures	consult list of 50 choices, in define.f
KS parameters	$\Delta t_{\rm cl},\ R_{\rm cl},\ \eta_{\rm U},\ \gamma_{\rm min}$
IMF	$\alpha,\ m_1,\ m_N,\ N_{\rm b},$ #20
Virial theorem	$Q_{\rm V} = 0.5$ for equilibrium
Primordial binaries	$a_{\rm max},\ e_0,\ m_1/m_2,\ a_{\rm min},$ #20

Numerical examples

$N = 1000,\ n_{\rm max} = 95,\ \eta_{\rm I} = 0.02,\ \eta_{\rm R} = 0.02,$

$S_0 = 0.3,\ \Delta T = 2,\ T_{\rm crit} = 100,$

$Q_{\rm E} = 2 \times 10^{-5},\ R_{\rm pc} = 1,\ \bar{m} = 0.5$

1, 2, 5, 7, 14, 16, 20, 23

$\Delta t_{\rm cl} = 10^{-4},\ R_{\rm cl} = 0.001,\ \eta_{\rm U} = 0.2,\ \gamma_{\rm min} = 10^{-6}$

$\alpha = 2.3,\ m_1 = 10.0,\ m_N = 0.2,$ #20 $= 1$

Modification of COMMON

(a) Constant size

Existing dummies	..., $XDUM(10)$, $NDUM(10)$
New variables	$XNEW(2)$, NEW
	..., $XNEW(2)$, $XDUM(8)$, NEW, $NDUM(9)$

(b) Enlargement

Increase COMMON	$COMMON/EXTRA/\ \ A(5),\ B,\ NEW(6)$
Add to MYDUMP	$REAL*4\ \ XNEW$
New COMMON	$COMMON/EXTRA/\ \ XNEW(18)$
Add READ/WRITE	..., $XNEW$

Decision-Making

Condition for special treatment	$\Delta t_i < \Delta t_{\rm cl}$
1. Two-body encounter	$i \leq N, \quad R < R_{\rm cl}, \quad \dot{R} \, < \, 0$
2. Chain regularization	$i > N, \quad a_{\rm out}\,(1 - e_{\rm out}) < 2\,a_{\rm in}$
3. Stable triple formation	$a_{\rm out}\,(1 - e_{\rm out}) > 3\,a_{\rm in}$
Apocentre test (#2 & #3)	$\dot{R}_0\,\dot{R} \, < \, 0, \quad R \, > \, a$
Case #1 algorithm	$\dfrac{P\,R^2}{m_1 + m_2} \, < \, 0.25, ...$
Case #2 algorithm	$R \, + \, d \, < \, R_{\rm cl}, ...$
Case #3 algorithm	$\dfrac{M_{\rm b} m_3}{2\,a_{\rm out}} > \dfrac{1}{2}\bar{m}\,V^2, ...$

Three-Body Stability Criterion

```
REAL*8 FUNCTION QSTAB(e,eout,zi,m1,m2,m3)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 m1,m2,m3
*
* Three-body stability function (Valtonen, AAS 2015).
*
* Inner and outer eccentricity: e & eout.
* Inclination in radians: zi.
* Masses: inner, m1  m2, outer, m3.
*
* Adopt 10,000 outer orbits for random walk time-scale.
ZN = 10000.0
zz = 1.0/6.0D0
*
F = 1.0 - 2.0*e/3.0*(1.0 - 0.5*e**2) - 0.3*cos(zi)*
&    (1.0 - 0.5*e + 2.0*cos(zi)*(1.0 - 2.5*e**1.5 - cos(zi)))
*
G = SQRT(m1/(m1 + m2))*(1.0 + m3/(m1 + m2))
*
QSTAB = 1.52*(SQRT(ZN)/(1.0 - eout))**zz*(F*G)**0.3333

RETURN
END
```

Close Encounters

Search for close companion $\Delta t_i < \Delta t_{\rm cl}$

Acceptance criterion $R < R_{\rm cl},\ \dot{R} < 0$

Form new two-body elements $\mathbf{R},\ \mathbf{B},\ P$

Introduce c.m. as fictitious particle $i = N + 1$

Initialize c.m. motion $\mathbf{F}_{\rm cm},\ \dot{\mathbf{F}}_{\rm cm},\ \Delta t_{\rm cm}$

Advance regularized solution up to $t_{\rm reg} = t$

Flyby termination $R > R_0$

Collision test $R < r_1^* + r_2^*$

Treat any other particles & c.m. up to t

Escape removal or solar accretion $r_i > R_{\rm esc},\quad e_i > 0.99$

Time-Steps

Taylor series $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,\Delta t + \frac{1}{2}\mathbf{F}_0^{(2)}\,\Delta t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\,\Delta t^3$

Two-body motion $\Delta t = R^{3/2}$

Natural time-step $\Delta t = \left(\eta \frac{|\mathbf{F}|}{|\mathbf{F}^{(2)}|}\right)^{1/2}, \quad \eta = 0.02$

General expression $\Delta t = \left(\eta \frac{|\mathbf{F}||\mathbf{F}^{(2)}|+|\mathbf{F}^{(1)}|^2}{|\mathbf{F}^{(1)}||\mathbf{F}^{(3)}|+|\mathbf{F}^{(2)}|^2}\right)^{1/2}$

Relative criterion Δt independent of mass

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Hierarchical levels $\mathcal{N}_k$ particles with steps Δt_k

Commensurability $\mathrm{Dmod}\ (t,\, \Delta t_i) = 0$

Program Control

Scheduling	Sorted list with $N^{1/2}$ members
Current time	$t = t_i + \Delta t_i, \quad i = NEXT(1)$
Next time	$T_{\rm min} = \min\,(t + \Delta t_j)$
Next block	All $t_j + \Delta t_j = T_{\rm min}$
Prediction	Full N or joint neighbour list
Irregular force	$\mathbf{F}_i, \mathbf{F}_i^{(1)}$ for n members
Regular force	$t_{\rm reg} + \Delta t_{\rm reg} = t_{\rm new}$
Continuity	Identical $\mathbf{F}_{\rm n}$, $\mathbf{F}_{\rm n}^{(1)}$ if no change
Strategy	Predicted coordinates for $\mathbf{F}_{\rm reg}$

Integration Parameters

$\eta_{\rm I}$	Time-step parameter for irregular force	0.02
$\eta_{\rm R}$	Time-step parameter for regular force	0.03
S_0	Initial radius of the neighbour sphere	0.30
$n_{\rm max}$	Maximum neighbour number	70
$\Delta t_{\rm adj}$	Time interval for energy check	2.0
$\Delta t_{\rm out}$	Time interval for main output	10.0
$Q_{\rm E}$	Tolerance for energy check	1×10^{-5}
$R_{\rm V}$	Virial cluster radius (length unit) in pc	2.0
$M_{\rm S}$	Mean stellar mass in solar units	0.5
$Q_{\rm vir}$	Virial theorem ratio $(T/\|U+2W\|)$	0.5
$\Delta t_{\rm cl}$	Time-step criterion for close encounters	1×10^{-4}
$R_{\rm cl}$	Distance criterion for KS regularization	1×10^{-3}
$\eta_{\rm U}$	Regularized time-step parameter	0.2
$h_{\rm hard}$	Energy per unit mass for hard binary	1.0
$\gamma_{\rm min}$	Limit for unperturbed KS motion	1×10^{-6}
$\gamma_{\rm max}$	Termination criterion for soft binaries	0.001

Optional Procedures

1	Manual common save on unit 1 at any time
2	Common save on unit 2 at output time or restart
3	Data bank on unit 3 with specified frequency
5	Different types of initial conditions
7	Output of Lagrangian radii
8	Primordial binaries (extra input required)
10	Two-body regularization diagnostics
14	External tidal force; open or globular clusters
15	Multiple regularization or hierarchical systems
16	Updating of regularization parameters $R_{\rm cl}$, $\Delta t_{\rm cl}$
17	Modification of $\eta_{\rm I}$ and $\eta_{\rm R}$ by tolerance $Q_{\rm E}$
19	Synthetic stellar evolution with mass loss
20	Different types of initial mass functions
23	Removal of distant escapers (isolated or tidal)
26	Slow-down of KS and/or chain regularization
27	Tidal circularization (sequential or continuous)
28	Magnetic braking and gravitational radiation
30	Chain regularization (with special diagnostics)

Basic Variables

$\mathbf{x}_0$	`X0`	Primary coordinates
$\mathbf{v}_0$	`X0DOT`	Primary velocity
$\mathbf{x}$	`X`	Prediction coordinates
$\mathbf{v}$	`XDOT`	Prediction velocity
$\mathbf{F}$	`F`	One half the total force (per unit mass)
$\mathbf{F}^{(1)}$	`FDOT`	One sixth the total force derivative
m	`BODY`	Particle mass (also initial mass m_0)
Δt	`STEP`	Irregular time-step
t_0	`T0`	Time of last irregular force
$\mathbf{F}_{\rm I}$	`FI`	Irregular force
$\mathbf{D}^1_{\rm I}$	`FIDOT`	First irregular force derivative
$\mathbf{D}^2_{\rm I}$	`D2`	Second irregular force derivative
$\mathbf{D}^3_{\rm I}$	`D3`	Third irregular force derivative
ΔT	`STEPR`	Regular time-step
T_0	`T0R`	Time of last regular forcex
$\mathbf{F}_{\rm R}$	`FR`	Regular force
$\mathbf{D}^1_{\rm R}$	`FRDOT`	First regular force derivative
$\mathbf{D}^2_{\rm R}$	`D2R`	Second regular force derivative
$\mathbf{D}^3_{\rm R}$	`D3R`	Third regular force derivative
$R_{\rm s}$	`RS`	Neighbour sphere radius
L	`LIST`	Neighbour and perturber list

KS Variables

$\mathbf{U_0}$	`U0`	Primary regularized coordinates
$\mathbf{U}$	`U`	Regularized prediction coordinates
$\mathbf{U}'$	`UDOT`	Regularized velocity
$\mathbf{F}_{\rm U}$	`FU`	One half the regularized force
$\mathbf{F}'_{\rm U}$	`FUDOT`	One sixth the regularized force derivative
$\mathbf{F}^{(2)}_{\rm U}$	`FUDOT2`	Second regularized force derivative
$\mathbf{F}^{(3)}_{\rm U}$	`FUDOT3`	Third regularized force derivative
h	`H`	Binding energy per unit reduced mass
h'	`HDOT`	First derivative of specific binding energy
$h^{(2)}$	`HDOT2`	Second derivative of binding energy
$h^{(3)}$	`HDOT3`	Third derivative of binding energy
$h^{(4)}$	`HDOT4`	Fourth derivative of binding energy
$\Delta\tau$	`DTAU`	Regularized time-step
$t^{(2)}$	`TDOT2`	Second regularized derivative of time
$t^{(3)}$	`TDOT3`	Third regularized derivative of time
R	`R`	Two-body separation
R_0	`R0`	Initial value of the two-body separation
γ	`GAMMA`	Relative perturbation

NBODY6 Output

Control line	$T \;\; Q_{\rm V} \;\; DE/E \;\; ETOT \;\; R_{\rm cl} \;\; \Delta t_{\rm min} \;\; DETOT \;\; CPU$
Main output	$T \;\; N \;\; NB \;\; KS \;\; NM \;\; MM \;\; NS \;\; NSTEPS \;\; DE/E$

Optional Procedures:

Cluster core	N^2 algorithm for core radius and density centre
Lagrangian radii	Percentile mass radii and half-mass radius
Error control	Automatic error check and restart from last time
Escape	Removal of distant members and table updates
Time offset	Rescaling of all global times
Events	Stellar types and energy partition
Binary analysis	Regularized binary histograms and energy budget
Binary data bank	Characteristic parameters for regularized binaries
HR diagram	Evolutionary state of single stars and binaries
General data bank	Detailed snapshots for data analysis

Hierarchy Detection

1. Identify the dominant and second perturber, j and p, from m/r^3
2. Form $\mathbf{r} \cdot \dot{\mathbf{r}}$ for m_j and $\gamma_3 = 2m_p/m_{\rm b}(R/r_p)^3$ with respect to m_i
3. Exit on positive radial velocity or $\gamma_3 > 10^4\gamma_{\rm min}$
4. Evaluate the minimum distance, $R_{\rm p} = a_{\rm out}(1 - e_{\rm out})$
5. Obtain actual perturbing force, $\mathbf{P}$, and $\tilde{\gamma} = |\mathbf{P}|r_j^2/(m_i + m_j)$
6. Define perturbation due to m_p by $\gamma_4 = 2(r_j/r_p)^3 m_p/(m_i + m_j)$
7. Combine semi-major axes if second binary is present; $j > N$
8. Check stability if $e_{\rm out} > 0.99$, $R_{\rm p} > 10R$, $\max\{\tilde{\gamma}, \gamma_4\} < \gamma_{\rm max}$
9. Continue checking if $r_j > \max\{3R_{\rm grav}, R_{\rm cl}\}$, $R_{\rm p} > 2a_{\rm in}(1 + e_{\rm in})$
10. Consider stability of two binaries if $j > N$ and $R_{\rm p} > a_{\rm in} + a_2$

Stellar Evolution

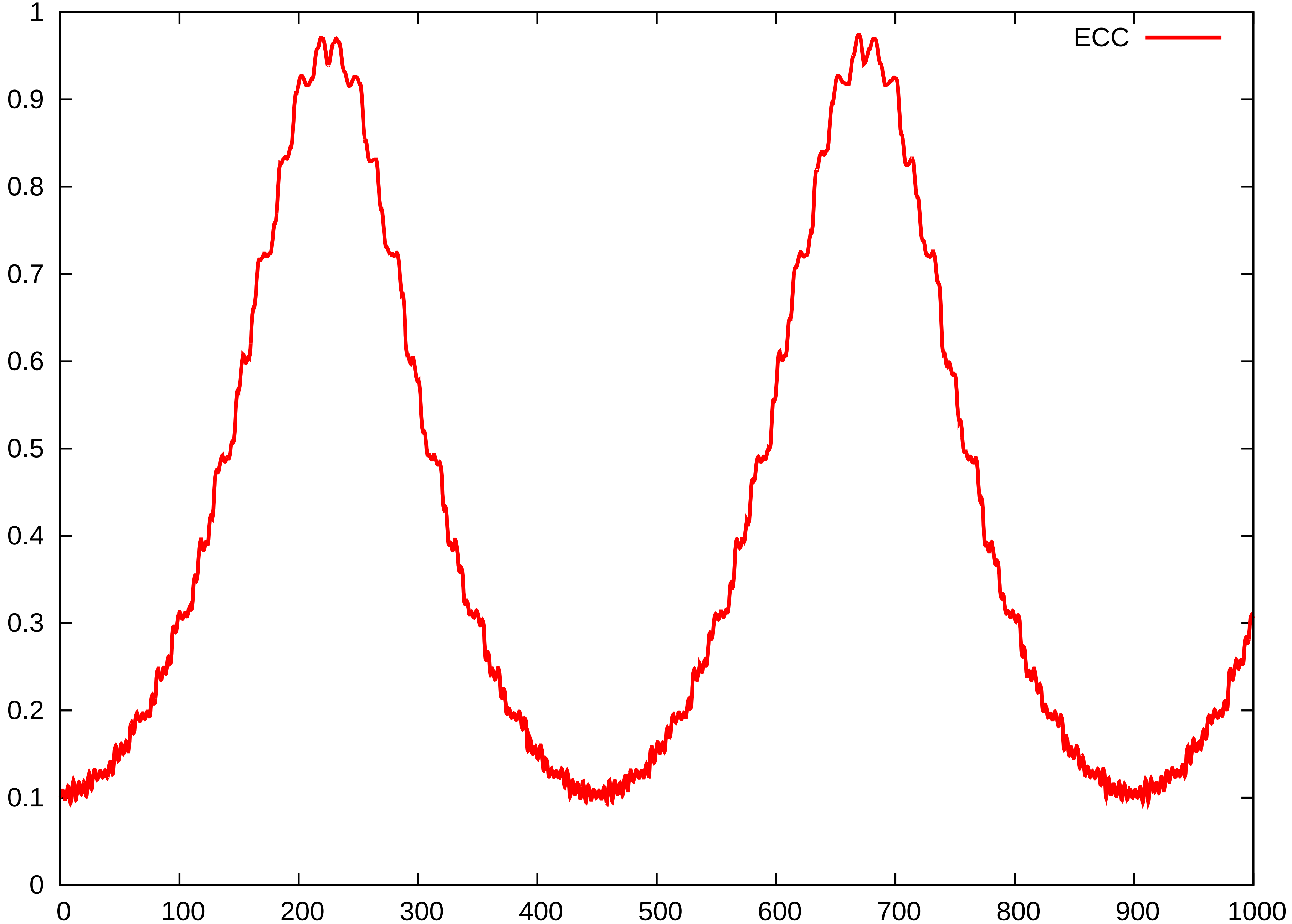
Stellar HR types	$K^* = 0, ..., 15$
Fast look-up (Pop I & II)	$r^*(t),\ m_c(t),\ L^*(t),\ K^*(t)$
Wind mass loss	$\dot{m} = -2 \times 10^{-13} r^* L^*/m$
Single stars	$\Delta m/m > 1\%$, new r^*
Updating times	$T_{\rm ev} = t + \min(\Delta t_{\rm ev},\ \Delta t_{\rm rem})$
Stellar rotation	$\Delta J_{\rm spin} = 2\Delta m r^2 \Omega_{\rm rot}/3$
White dwarfs	cooling curves, $\Delta t_{\rm ev} = 10^6\,{\rm yr}$
Supernova outburst	$m_{\rm c} > m_{\rm chandra} \Rightarrow$ SN
NS velocity kick	$v >> v_\infty \sim 2\,km/s$
Binary mass loss	$m\,a = {\rm const}$
Synthetic HR diagram	binaries and single stars
Energy conservation	$\Delta E = \Delta m\,(\frac{1}{2} v^2 + \Phi)$

Binary Processes

Tidal circularization	$a\,(1 - e^2) = \text{const} \quad \Rightarrow \quad \dot{a} < 0$
Roche-lobe mass transfer	$r^* > r_{\rm RL}, \quad \Delta m_2 = -f \Delta m_1$
Common envelope evolution	$m_c > 0, \quad$ MS + giant
Magnetic braking	$\dot{a}_{\rm MB} \propto a^{-4}$
Gravitational radiation	$\dot{a}_{\rm GR} \propto a^{-3}$
Spin-orbit coupling	$J_{\rm tot} = J_{\rm orb} + J_{\rm spin}$
Stellar collisions	$a\,(1 - e) < 0.75\,(r_1^* + r_2^*)$
Blue stragglers	mass transfer or MS collisions
Cataclysmic variables	WD + giant
X-ray objects	WD + MS, NS + MS
Doubly degenerates	WD + WD, $P \simeq 10$ mins
Type Ia supernova	WD – WD collision or inspiral

Physical Collisions

Simple definition	$R_{\rm coll} = \frac{3}{4}(r_1^* + r_2^*)$
Two-body encounter	KS regularization
Pericentre condition	$R_0' \, R' < 0, \quad R < a$
Pericentre determination	$\Delta t_{\rm peri}$ from Kepler's equation
Predict $\mathbf{R}_{peri}$ or iterate	$d\tau_0 = \frac{\Delta t_{peri}}{R}$, Newton-Raphson
Implement collision	$m_{\rm cm} = m_1 + m_2, \quad r_{\rm cm} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$
Mass loss	$\Delta m = f(K_1^*, K_2^*)$
Initialize single body	$\mathbf{F}_1, \, \dot{\mathbf{F}}_1, \, \Delta t_1$
Compact subsystem	$\dot{R} \simeq 0$ by iteration
Transformation	$\mathbf{Q}, \, \mathbf{P} \Rightarrow \mathbf{r}, \, \dot{\mathbf{r}}$
New chain construction	$N_{\rm ch} \Rightarrow N_{\rm ch} - 1, \quad E_{\rm coll} = E_{\rm ch} - \mathcal{V}$



ECC
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
100
200
300
400
500
600
700
800
900
1000

PN Energy Loss

Equation of motion

$$\mathbf{a} = \mathbf{a}_N + \mathbf{a}_{\mathrm{pn}} = -\frac{m}{r^3}\mathbf{r} + \frac{m}{r^2 c^5}\left(A\frac{\mathbf{r}}{r} + B\mathbf{v}\right)$$

Energy change

$$\Delta E_{\mathrm{GR}} = \mu\left(\mathbf{a}_{\mathrm{pn}} \cdot \mathbf{v}\,\Delta t + \frac{1}{2}(\mathbf{a}_{\mathbf{pn}} \cdot \mathbf{a} + \dot{\mathbf{a}}_{\mathrm{pn}} \cdot \mathbf{v})\Delta t^2\right)$$

Derivatives $\dot{A},\quad \dot{B},\quad$ save $\dot{\epsilon}_0,\quad \ddot{\epsilon}_0$

Hermite solution Second term as $\ddot{\epsilon}$

Fourth-order solution

$$\Delta\epsilon = \frac{1}{2}(\dot{\epsilon}_0 + \dot{\epsilon})\,\Delta t + \frac{1}{12}(\ddot{\epsilon}_0 - \ddot{\epsilon})\,\Delta t^2$$

Comparison Agreement with PN 2.5 of Peters 1964

Getting Started

1. Download code `nbody6.tar.gz`
2. Unzip `gunzip nbody6.tar.gz`
3. Extract files `tar xvf nbody6.tar`
4. Check `params.h` `NMAX, LMAX, KMAX, MMAX`
5. Compile the code `make nbody6`
6. Create run directory `mkdir Run`
7. Run test input `time nbody6 <input >output &`
8. Profiling Makefile with -O3 -pg
9. Performance data `gprof nbody6 gmon.out -p >OUT`

NBODY6 Input File

1 20.0

1000 1 5 50000 95 1

0.02 0.02 0.3 2.0 10.0 100.0 1.0D-04 1.0 0.5

0 0 0 0 1 0 1 0 0 0

0 0 0 1 1 1 0 1 3 4

1 0 2 0 0 0 0 0 0 2

0 0 0 0 0 0 0 0 0 1

0 0 0 0 0 0 0 0 0 0

4.0D-05 4.0D-04 0.2 1.0 1.0D-06 0.001

2.3 10.0 0.2 0 0 0.02 0.0 100.0

0.5 0 0 0 0.125

KSTART TCOMP

N NFIX NCRIT NRAND NNBMAX NRUN

ETAI ETAR RS0 DTADJ DELTAT TCRIT QE RBAR ZMBAR

OPTIONS (50)

DTMIN RMIN ETAU ECLOSE GMIN GMAX

ALPHA BODY1 BODYN NBIN0 NHI0 ZMET EPOCH0 DTPLOT

Q 0 0 0 SMAX

Problem Shooting

Illegal term:	Recompile with debugger
Looping:	Determine last block-step in INTGRT
Problem ID:	Counters NSTEPU, NSTEPI, NSTEPR
Small steps:	Sudden shrinkage of STEPR or STEP
Force errors:	Difference FI & FIRR on NNB=const
Energy errors:	Restart with smaller output interval
Energy errors:	High velocity search in ADJUST
Energy errors:	Monitor EBIN, ECOLL, EMERGE
Accuracy test:	Reduce step parameters $\eta_U,\ \eta_I,\ \eta_R$
Elimination:	Switch off procedures (chain, mass loss)
Diagnostics:	Activate options for more information
Neighbours:	Consider the neighbour strategy
Stability:	Long-lived triples outside boundary
Trap:	Consistency check of neighbour lists
Trap:	Binary c.m. consistency check