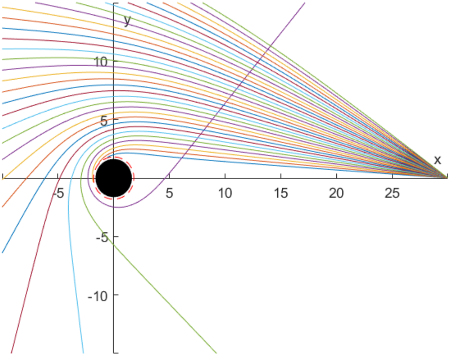
Light trajectories near compact relativistic sources surrounded by media of dispersive and refractive properties



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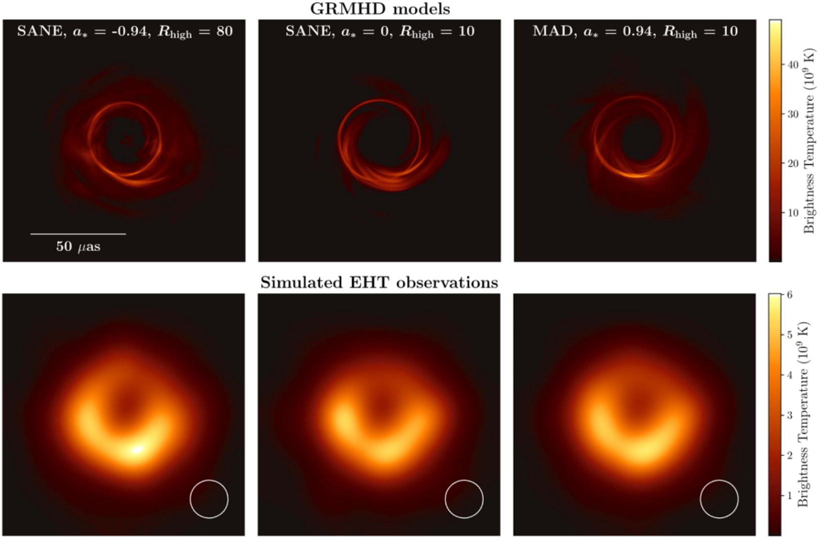
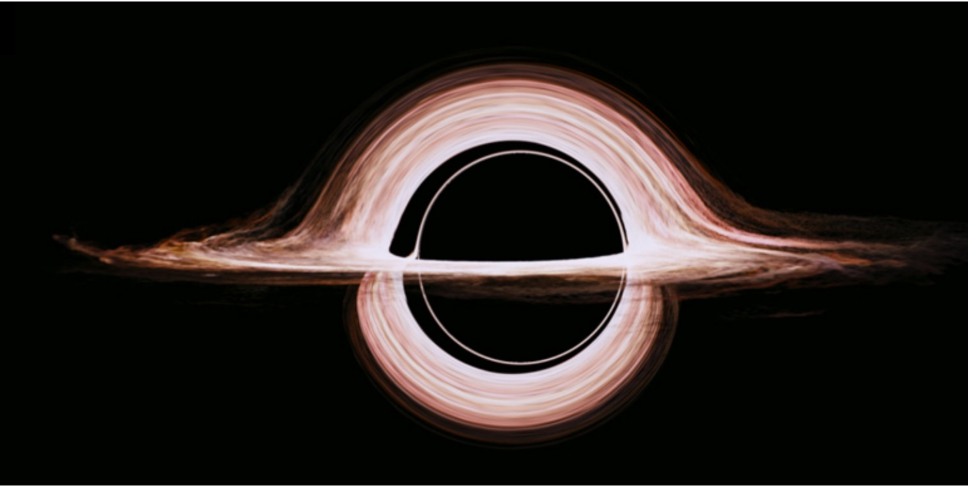
- Motivation
- Ray Propagation in Plasma
- Deflection Angle
- Moving Medium and Spherically Symmetric Object
- Static Medium and Axially Symmetric Object
- Ray Trajectories in the Kerr and HT Metrics
- Plasma Effect
- Quadrupole Moment Effect
- Conclusions

Motivation



EHT Collaboration I (2019)



James et al. (2015)

Relativistic geometrical optics seen as an approximation of the Maxwell equations can be adapted if the following is true (cf. Bičák & Hadrava, 1975)

- typical wavelength λ_t of considered waves is short in comparison with the typical scales on which the properties of the medium (i.e., the refractive index, the velocity) vary,
- the waves are locally monochromatic, i.e., the variation scales of wave properties (the amplitude, wavelength, polarization) are large in comparison with λ_t,
- the characteristic curvature radius of the spacetime is much larger than λ_t,
- the medium varies negligibly over one typical wave period.

Typically a non-dispersive isotropic medium is assumed, but this approach can serve also in dispersive and anisotropic media.

It is useful to apply Hamiltonian in the form[1]

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[g^{\beta\delta}p_\beta p_\delta - (n^2-1)(p_\gamma V^\gamma)^2\right], \qquad (1)$$

where $g^{\beta\delta}$ is the spacetime metric[2], x^α are the spacetime coordinates, p_α denotes the wave vector, V^α is the medium velocity, and

$$p_\gamma V^\gamma = -\omega(x^\alpha, p_\alpha)$$

defines the wave frequency.

Hamilton equations of motion are

$$\frac{dx^\alpha}{d\lambda} = \frac{\partial \mathcal{H}}{\partial p_\alpha}, \quad \frac{dp_\alpha}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\alpha}.$$

[1] Synge (1960)   [2] $\alpha, \beta = 0, 1, 2, 3$; $t = 0$

The medium is described by n and V^α and it is generally characterized as

Refractive
$n \neq 1$

Dispersive
$n = n(x^\alpha, \omega(x^\alpha, p_\alpha))$

In given calculations, *cold plasma* is typically assumed, while the applied formalism is more general. General medium WITHOUT motion was studied by Tsupko (2021).

$$\frac{\omega}{k} \gg v_{th}$$

Taking

$$n^2 = 1 - \frac{\omega_{pl}^2(x^\alpha)}{\omega^2(x^\alpha)},$$

where

$$\omega_{pl}^2(x^\alpha) = \frac{e^2}{\epsilon_0 m_e} N(x^\alpha)$$

is the electron plasma frequency, and plugging into (1) gives

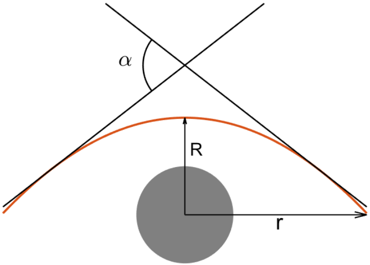
$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[g^{\beta\delta} p_\beta p_\delta + \omega_{pl}^2(x^\alpha)\right].$$

Light propagation is hence INDEPENDENT of the medium velocity which appears only in $\omega(x^\alpha)$ which vanishes.
Note that wave frequency must obey $\omega(x^\alpha) > \omega_{pl}(x^\alpha)$.

Deflection Angle Definition

$$\Delta\varphi = 2\int_R^\infty \mathcal{F}(r)dr$$

$\mathcal{F}$ given as $\frac{d\varphi}{dr} = \mathcal{F}(r)$



$$\alpha = \Delta\varphi - \pi$$

Notice difficulties when defining the deflection angle OUTSIDE the equatorial plane.

To describe a gravitating object, a spherically symmetric and static metric is chosen.[3] It generally reads

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + C(r)(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2),$$

where $A(r)$, $B(r)$, and $C(r)$ are positive functions. Additionally, asymptotic flatness is requested, hence for $r \to \infty$

$$A(r) \to 1, \quad B(r) \to 1, \quad \frac{C(r)}{r^2} \to 1.$$

Furthermore, the medium is assumed to be spherically symmetric as well, i.e., $n = n(r, \omega(r))$.

[3] Based on Bezděková et al. (2024).

The medium which surrounds the compact object can rotate, which defines its 4-velocity as

$$V^\alpha = (V^0, 0, 0, V^\varphi), \quad V^\varphi = f(r).$$

The medium velocity is independent of t and the component p_0 hence is a constant of motion denoted further as $-\omega_0$. Component V^0 derived from the 4-velocity normalization reads

$$V^0 = \sqrt{\frac{1 + C(r) f^2(r)}{A(r)}}.$$

The general form of photon frequency in this medium yields

$$\omega(p_\varphi, r) = -p_0 V^0(r) - p_\varphi V^\varphi = \omega_0 V^0(r) - p_\varphi f(r).$$

Rotating Medium

Assuming the equatorial plane ($p_\vartheta = 0$), the Hamiltonian reads

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[\frac{p_r^2}{B(r)} + \frac{p_\varphi^2}{C(r)} - \frac{\omega_0^2}{A(r)} + w(r, \omega(p_\varphi, r))\right],$$

where[4]

$$w(r, \omega(p_\varphi, r)) = -[n^2(r, \omega(p_\varphi, r)) - 1]\omega^2(p_\varphi, r).$$

Equation of motion for component r yields

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{B(r)},$$

and for φ it returns

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial p_\varphi} = \frac{p_\varphi}{C(r)} + \frac{1}{2}\frac{\partial w}{\partial \omega}\frac{\partial \omega(p_\varphi, r)}{\partial p_\varphi}.$$

[4] In cold plasma $w = \omega_{pl}^2(r)$.

The refractive index can generally be written as

$$n^2(r,\omega(p_\varphi,r)) = a_0(r) + \frac{a_1(r)}{\omega(p_\varphi,r)} + \frac{a_2(r)}{\omega^2(p_\varphi,r)}.$$

It turns out to be convenient to express the Hamiltonian as

$$\mathcal{H}(x^\alpha,p_\alpha) = \frac{1}{2}\left[\mathcal{A}_\varphi(r)p_\varphi^2 + 2\mathcal{B}_\varphi(r)p_\varphi + \mathcal{C}_\varphi(r,p_r)\right],$$

where $\mathcal{A}_\varphi(r)$, $\mathcal{B}_\varphi(r)$, $\mathcal{C}_\varphi(r,p_r)$ are additional functions of r, eventually of p_r. Without the loss of generality it can be assumed that

$$\mathcal{C}_\varphi(r,p_r) = \frac{p_r^2}{B(r)} + \mathcal{C}_{\varphi 1}(r).$$

Applying the general procedure outlined above leads to the form of the deflection angle

$$\alpha = 2 \int_R^\infty \sqrt{B(r)\mathcal{A}_\varphi(r)} \left[\frac{h^2(r)}{\left(\frac{\mathcal{B}_\varphi(r)}{\mathcal{A}_\varphi(r)} - \frac{\mathcal{B}_\varphi(R)}{\mathcal{A}_\varphi(R)} \pm h(R) \right)^2} - 1 \right]^{-1/2} dr - \pi,$$

where

$$h^2(r) = \frac{1}{\mathcal{A}_\varphi^2(r)} \left(\mathcal{B}_\varphi^2(r) - \mathcal{A}_\varphi(r)\mathcal{C}_{\varphi 1}(r) \right).$$

Note that the functions $\mathcal{A}_\varphi(r)$, $\mathcal{B}_\varphi(r)$, $\mathcal{C}_{\varphi 1}(r)$ are generally also functions of constants of motion ω_0 and p_φ.

$$ds^2 = -A(r,\vartheta)dt^2 + B(r,\vartheta)dr^2 + 2P(r,\vartheta)dt d\varphi + C(r,\vartheta)d\varphi^2 + D(r,\vartheta)d\vartheta^2$$

$B(r,\vartheta) > 0, \quad D(r,\vartheta) > 0, \quad A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta) > 0$
+ dependence on other parameters, e.g., angular momentum a

The corresponding Hamiltonian reads[5]

$$\mathcal{H}(x^\alpha, p_\alpha) = \frac{1}{2}\left[\frac{p_r^2}{B(r,\vartheta)} + \frac{p_\vartheta^2}{D(r,\vartheta)} + \frac{p_\varphi^2 A(r,\vartheta) - \omega_0^2 C(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} - \frac{2\omega_0 p_\varphi P(r,\vartheta)}{A(r,\vartheta)C(r,\vartheta) + P^2(r,\vartheta)} + w(r,\vartheta,\omega(r,\vartheta))\right].$$

Further, the equatorial plane is considered, i.e., $\vartheta = \pi/2$, $p_\vartheta = 0$.

[5]For details, see Bezděková & Bičák (2023).

The medium is regarded as *static*, i.e.,

$$V^\alpha = (V^0, 0, 0, 0),$$

which leads to

$$V^0 = \frac{1}{\sqrt{A(r)}},$$

and then

$$w(r, \omega(r)) = -(n^2 - 1)\frac{\omega_0^2}{A(r)}.$$

Moreover, the refractive index in cold plasma yields

$$n^2 = 1 - \frac{\omega_{pl}^2(r)}{\omega^2(r)} = 1 - \frac{\omega_{pl}^2(r)}{\omega_0^2}A(r).$$

Under the given conditions and after performing the same procedure as before, one gets

$$\alpha = 2\int_R^\infty \sqrt{\frac{A(r)B(r)}{A(r)C(r)+P^2(r)}}\left(\frac{h^2(r)}{\left(\frac{P(R)}{A(R)}-\frac{P(r)}{A(r)}\pm h(R)\right)^2}-1\right)^{-1/2} dr - \pi,$$

where

$$h^2(r) = \frac{C(r)}{A(r)}n^2 + \frac{P^2(r)}{A^2(r)}.$$

This is a general formula for the deflection angle in an axially symmetric stationary spacetime with an arbitrary static dispersive medium given by its refractive index n.

Notice a formal correspondence with the deflection angle for a spherically symmetric object in the rotating medium.

For the Kerr metric (in the equatorial plane) it holds

$$A(r) = 1 - \frac{2M}{r}, \qquad B(r) = \frac{r^2}{r^2 - 2Mr + a^2},$$
$$C(r) = r^2 + a^2 + \frac{2Ma^2}{r}, \quad P(r) = \frac{-2Ma}{r}$$

The deflection angle in cold plasma is[6]

$$\alpha = 2\int_R^\infty \frac{\sqrt{r(r-2M)}}{r^2 - 2Mr + a^2} \left(\frac{\frac{r(r^2-2Mr+a^2)}{r-2M}\left(\frac{r}{r-2M} - \frac{\omega_p^2(r)}{\omega_0^2}\right)}{\left(\frac{2Ma}{r-2M} - \frac{2Ma}{R-2M} \pm h(R)\right)^2} - 1 \right)^{-1/2} dr - \pi,$$

where

$$h(R) = \sqrt{\frac{R(R^2 - 2MR + a^2)}{R - 2M}\left(\frac{R}{R-2M} - \frac{\omega_p^2(R)}{\omega_0^2}\right)}.$$

[6] Cf. Perlick (2000).

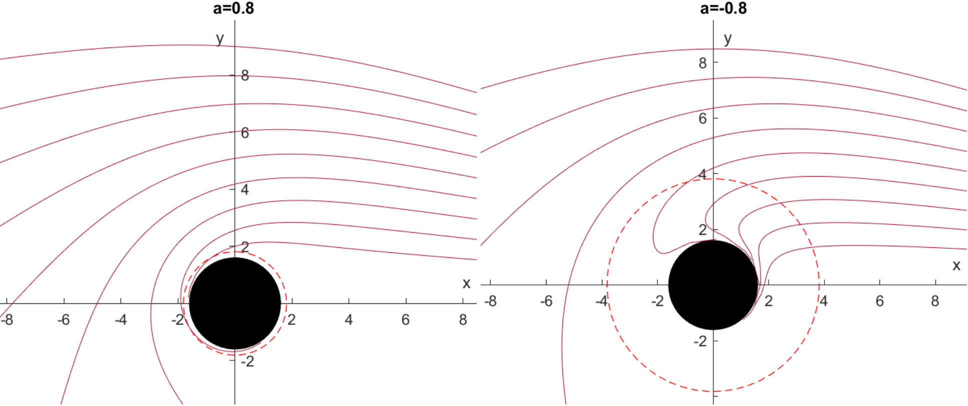


Figure 1: Ray trajectories in the vicinity of a Kerr black hole when the light either co-rotates (left) or counter-rotates (right) with the black hole.

Hartle-Thorne (HT) Metric[7]

$$
\begin{aligned}
ds^2 = &-\left(1-\frac{2M}{r}+\frac{2J^2}{r^4}\right)\left\{1+2P_2(\cos\vartheta)\left[\frac{J^2}{Mr^3}\left(1+\frac{M}{r}\right)\right.\right.\\
&+\left.\left.\frac{5}{8}\frac{Q-J^2/M}{M^3}Q_2^2\left(\frac{r}{M}-1\right)\right]\right\}dt^2\\
&+\left(1-\frac{2M}{r}+\frac{2J^2}{r^4}\right)^{-1}\left\{1-2P_2(\cos\vartheta)\left[\frac{J^2}{Mr^3}\left(1-\frac{5M}{r}\right)\right.\right.\\
&+\left.\left.\frac{5}{8}\frac{Q-J^2/M}{M^3}Q_2^2\left(\frac{r}{M}-1\right)\right]\right\}dr^2\\
&+r^2\left\{1+2P_2(\cos\vartheta)\left[-\frac{J^2}{Mr^3}\left(1+\frac{2M}{r}\right)\right.\right.\\
&+\left.\left.\frac{5}{8}\frac{Q-J^2/M}{M^3}\left\langle\frac{2M}{\sqrt{r(r-2M)}}Q_2^1\left(\frac{r}{M}-1\right)-Q_2^2\left(\frac{r}{M}-1\right)\right\rangle\right]\right\}\\
&\times\left\{d\vartheta^2+\sin^2\vartheta\left(d\varphi-\frac{2J}{r^3}dt\right)^2\right\}
\end{aligned}
$$

[7] Hartle & Thorne (1968)

M – total mass
J – total angular momentum
Q – quadrupole moment
$Q_2^1(x)$, $Q_2^2(x)$ – the associated Legendre functions of the second kind, $P_2(\cos\vartheta)$ – the Legendre polynomial of the second order

Let us introduce a compact notation for the following dimensionless quantities:

$$A_1 = 1 - \frac{2M}{r} + \frac{2J^2}{r^4}, \qquad j = \frac{J^2}{Mr^3},$$

$$K = \frac{5}{8}\frac{Q - J^2/M}{M^3}, \qquad j_1 = \frac{2J}{r^2},$$

$$Q_2^1 = Q_2^1\left(\frac{r}{M} - 1\right), \qquad Q_2^2 = Q_2^2\left(\frac{r}{M} - 1\right).$$

Kerr metric – $Q = J^2/M$ ($K = 0$) and $J = -Ma$
To derive the corresponding form of the Kerr metric from the HT metric, one has to apply the transformation (from the Boyer-Lindquist coordinates)

$$r \to r\left[1 - \frac{a^2}{2r^2}\left(\left(1+\frac{2M}{r}\right)\left(1-\frac{M}{r}\right) - \cos^2\vartheta\left(1-\frac{2M}{r}\right)\left(1+\frac{3M}{r}\right)\right)\right],$$

$$\vartheta \to \vartheta - \frac{a^2}{2r^2}\sin\vartheta\cos\vartheta\left(1+\frac{2M}{r}\right).$$

Deflection Angle in the HT Metric

The deflection angle in the HT metric in the cold plasma gives

$$\alpha_{HT} = 2\int_R^\infty f_{HT}(r)dr - \pi,$$

where

$$f_{HT}(r) = \sqrt{\frac{AB_0}{ACP_0 - K\frac{2Mr^2}{\sqrt{r(r-2M)}}Q_2^1}}$$

$$\times\left(\frac{\left(ACP_0 - K\frac{2Mr^2}{\sqrt{r(r-2M)}}Q_2^1\right)\left(1-\frac{\omega_p^2(r)}{\omega_0^2}\left(A_0 - KQ_2^2\right)\right)}{\left(A_0 - KQ_2^2\right)^2\left(PA_0(R) - PA_0(r) \pm h(R)\right)^2} - 1\right)^{-1/2},$$

with

$$A_0 = 1 - \frac{2M}{r} - j,$$
$$AB_0 = 1,$$
$$ACP_0 = r^2 \left(1 - \frac{2M}{r}\right),$$
$$PA_0 = -\frac{2J}{r} \left(1 + \frac{2M}{r}\right),$$
$$h^2(R) = \frac{R^2 \left(1 - \frac{2M}{R}\right)}{\left(1 - \frac{2M}{R} - j\right)^2} \left(1 - \frac{\omega_p^2(R)}{\omega_0^2} \left(1 - \frac{2M}{R} - j\right)\right).$$

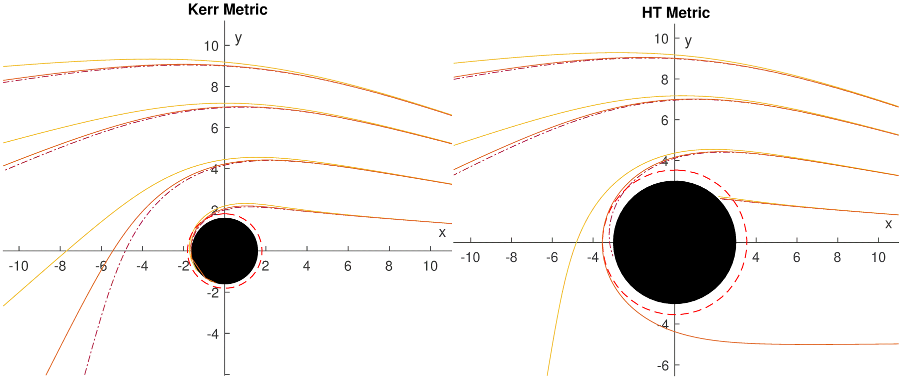


Figure 2: Ray trajectories in the vicinity of the black hole in vacuum (dash-dotted lines) and cold plasma cases (solid lines). The plasma frequency was characterized as $10\omega_0^2\left(\frac{M}{r}\right)^k$ with $k = 5/2$ (yellow lines) or $k = 7/2$ (orange lines). The dashed red circles show position of the radius of the circular photon orbit. The impact parameters of the rays defined as p_φ/ω_0 are $2, 5, 8, 10$.

Quadrupole Moment Effect

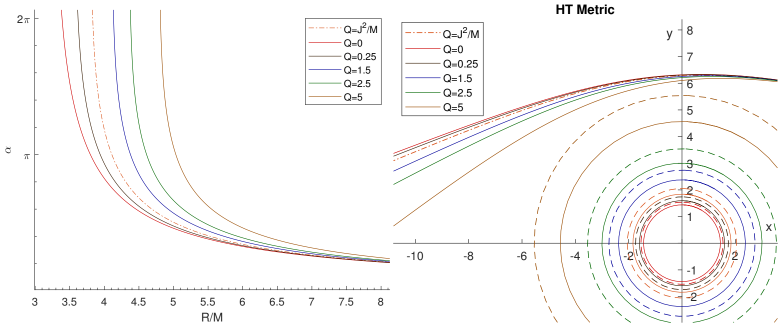


Figure 3: Effect of the quadrupole moment in the HT metric with J = 0.8 on the deflection angle and ray trajectories in plasma. Cold plasma defined with $k = 5/2$ was assumed. Ray impact parameter is equal to 7.

There is a general straightforward relation between impact parameter $b = \frac{p_\varphi}{\omega_0}$ and minimal radial distance R given as[8]

$$b = \frac{P(R)}{A(R)} \pm h(R) = \frac{P(R)}{A(R)} \pm n\sqrt{\frac{C(R)}{A(R)} + \frac{P^2(R)}{A^2(R)}}.$$

E.g., for the HT metric in vacuum when $K = 0$ it reads

$$b = \frac{R}{\sqrt{1 - \frac{2M}{R} + \frac{4Ma}{R^2} - \frac{4M^2a}{R^3} - \frac{2Ma^2}{R^3}}}.$$

[8] See Bezděková & Bičák (2023).

Conclusions

- The general approach how to obtain the deflection angle for a compact object surrounded by a refractive medium in the frame of Hamiltonian formalism was outlined.
- Rotating medium was discussed around a spherically symmetric object.
- Static medium was considered to surround an axially symmetric stationary object.
- Ray trajectories in the vicinity of objects described by the Kerr and HT metrics showed that the plasma presence causes the rays are less bent than in vacuum.
- Presence of the quadrupole moment can lead to both increase and decrease of the light bending in the vicinity of a gravitating object.
- The general relationship between b and R can be found.

1. B. Bezděková, V. Perlick, and J. Bičák. Light propagation in a plasma on an axially symmetric and stationary spacetime: Separability of the Hamilton–Jacobi equation and shadow. *J. Math. Phys.*, 63(9):092501, 2022.
2. B. Bezděková and J. Bičák. Light deflection in plasma in the Hartle-Thorne metric and in other axisymmetric spacetimes with a quadrupole moment. *Phys. Rev. D*, 108(8), 084043, 2023.
3. B. Bezděková, O. Tsupko, and C. Pfeifer. Deflection of light rays in a moving medium around a spherically symmetric gravitating object. *Phys. Rev. D*, 109(12), 124024, 2024.