Chapter 1

Basics of statistical physics

$$\dot{\boldsymbol{r}}_i = \frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{p}_i} \quad \text{and} \quad \dot{\boldsymbol{p}}_i = -\frac{\partial \mathcal{H}(\boldsymbol{q})}{\partial \boldsymbol{r}_i}, \quad \forall i = 1 \dots N, \tag{1.1}$$

1.1 Liouville's theorem

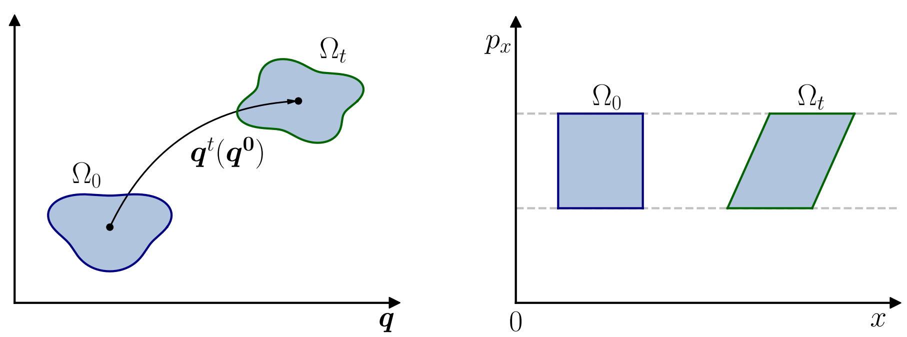
$$\int_{\Omega_0} \mathrm{d}^{6\mathrm{N}} q = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}} q = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}} q^t \,. \tag{1.2}$$

$$\int_M f(\boldsymbol{y}) \mathrm{d}^N \boldsymbol{y} = \int_{\phi^{-1}(M)} f\left(\phi(\boldsymbol{x})\right) \left|\det \frac{\mathrm{D}\phi}{\mathrm{D}x}\right| \mathrm{d}^N \boldsymbol{x} \,. \tag{1.3}$$

$$\frac{\mathrm{D}\phi}{\mathrm{D}x} = \begin{bmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_2}{\partial x_1} & \cdots & \frac{\partial \phi_N}{\partial x_1} \\ \frac{\partial \phi_1}{\partial x_2} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \\ \vdots & & & \ddots \end{bmatrix} . \tag{1.4}$$

$$\int_{\Omega_t} \mathrm{d}^{6\mathrm{N}} \boldsymbol{q} = \int_{\Omega_t} \mathrm{d}^{6\mathrm{N}} \boldsymbol{q}^t = \int_{\Omega_0} \left|\det \frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \mathrm{d}^{6N} \boldsymbol{q}^{\boldsymbol{0}} \,. \tag{1.5}$$

$$\int_{\Omega_t} 1 \mathrm{d}^{6\mathrm{N}} \boldsymbol{q} = \int_{\Omega_0} 1 \mathrm{d}^{6\mathrm{N}} \boldsymbol{q} \,. \tag{1.6}$$



$$J^t \equiv \left|\det\frac{\mathrm{D}q^t}{\mathrm{D}q^0}\right| \ , \tag{1.7}$$

$$\frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \frac{\partial}{\partial q^0_j}\frac{\mathrm{d}q^t_i}{\mathrm{d}t} = \frac{\partial \dot{q}^t_i}{\partial q^0_j} = \sum_{k=1}^{6N}\frac{\partial \dot{q}^t_i}{\partial q^t_k}\frac{\partial q^t_k}{\partial q^0_j} = \sum_{k=1}^{6N}\frac{\partial \dot{q}^t_i}{\partial q^t_k}J^t_{kj} \ , \tag{1.8}$$

$$J^t_{kj} = \frac{\partial q^t_k}{\partial q^0_j}. \tag{1.9}$$

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = \sum_{i,j}\frac{\partial J^t}{\partial J^t_{ij}}\frac{\mathrm{d}J^t_{ij}}{\mathrm{d}t} = \sum_{i,j,k}\frac{\partial J^t}{\partial J^t_{ij}}\,J^t_{kj}\,\frac{\partial \dot{q}^t_i}{\partial q^t_k} \ . \tag{1.10}$$

$$\sum_j \frac{\partial J^t}{\partial J^t_{ij}}J^t_{kj} = \delta_{ik}J^t \tag{1.11}$$

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t\sum_{i=1}^{6N}\frac{\partial \dot{q}^t_i}{\partial q^t_i} \ . \tag{1.12}$$

$$\frac{\mathrm{d}J^t}{\mathrm{d}t} = J^t\sum_{i=1}^{3\mathrm{N}}\left(\frac{\partial \dot{r}^t_i}{\partial r^t_i}+\frac{\partial \dot{p}^t_i}{\partial p^t_i}\right) = J^t\sum_{i=1}^{3\mathrm{N}}\left(\frac{\partial^2\mathcal{H}}{\partial r^t_i\partial p^t_i}-\frac{\partial^2\mathcal{H}}{\partial p^t_i\partial r^t_i}\right) \tag{1.13}$$

$$B = \frac{1}{\det A}\mathrm{adj}A, \tag{1.14}$$

$$(\mathrm{adj}A)^{ij} \equiv (-1)^{i+j}\det(A^{ji}). \tag{1.15}$$

$$b_{ij} = \frac{(-1)^{i+j}}{a}\det\left(A^{ji}\right),$$

$$a = \sum_i (-1)^{i+j}a_{ij}\det A^{ij}.$$

$$\frac{\partial a}{\partial a_{ij}} = (-1)^{i+j}\det A^{ij} = ab^{ji}. \tag{1.16}$$

$$\sum_j \frac{\partial a}{\partial a_{ij}}a_{kj} = \sum_j ab^{ji}a_{kj} = a\sum_j b^{ji}a_{kj} = a(BA)_{ki} = a\delta_{ik}, \tag{1.17}$$

1.2 Statistical description of a physical system

$$\int_{-\infty}^{+\infty} D_\mathrm{N}\mathrm{d}\omega = \mathcal{C}, \tag{1.18}$$

$$\frac{\partial}{\partial t}\int_\Omega D_\mathrm{N}\,\mathrm{d}\omega = -\oint_{\partial\Omega} D_\mathrm{N}\,\dot{\boldsymbol{q}}\cdot\mathrm{d}\boldsymbol{\Sigma} \tag{1.19}$$

$$\int_\Omega \frac{\partial D_\mathrm{N}}{\partial t}\,\mathrm{d}\omega = -\int_\Omega \nabla\cdot\left(D_\mathrm{N}\dot{\boldsymbol{q}}\right)\mathrm{d}\omega \ . \tag{1.20}$$

$$\frac{\partial D_{\mathrm{N}}}{\partial t} = -\nabla \cdot (D_{\mathrm{N}}\, \dot{\boldsymbol{q}}) \ . \tag{1.21}$$

$$\frac{\partial D_{\mathrm{N}}}{\partial t} = -\sum_{i=1}^{6N} \left[\frac{\partial D_{\mathrm{N}}}{\partial q_i} \dot{q}_i + D_{\mathrm{N}} \frac{\partial \dot{q}_i}{\partial q_i} \right] = -\sum_{i=1}^{3N} \left[\frac{\partial D_{\mathrm{N}}}{\partial r_i} \frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial D_{\mathrm{N}}}{\partial p_i} \frac{\partial \mathcal{H}}{\partial r_i} + D_{\mathrm{N}} \left(\frac{\partial \mathcal{H}}{\partial r_i \partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i \partial r_i} \right) \right] \ . \tag{1.22}$$

$$\frac{\partial D_{\mathrm{N}}}{\partial t} = \{\mathcal{H}, D_{\mathrm{N}}\}_{\mathrm{P}} \ . \tag{1.23}$$

$$f^1(\boldsymbol{r}_1, \boldsymbol{p}_1) = \int D_{\mathrm{N}}(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N.$$

$$f^2(\boldsymbol{r}_2, \boldsymbol{p}_2) = \int D_{\mathrm{N}}(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_N, \boldsymbol{p}_N) \mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{p}_1 \mathrm{d}^3\boldsymbol{r}_3 \mathrm{d}^3\boldsymbol{p}_3 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N$$

$$f(\boldsymbol{r}, \boldsymbol{p}) = \sum_{i=1}^{N} f^i(\boldsymbol{r}, \boldsymbol{p}) = N f^1(\boldsymbol{r}, \boldsymbol{p}) = N f^2(\boldsymbol{r}, \boldsymbol{p}) = \ldots .$$

1.3 Reduced distribution functions

$$f(\boldsymbol{r}, \boldsymbol{p}) \equiv \int D_{\mathrm{N}}(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}_2, \boldsymbol{p}_2, \ldots, \boldsymbol{r}_{\mathrm{N}}, \boldsymbol{p}_{\mathrm{N}})\, \mathrm{d}\tau_{\mathrm{N}-1} \ , \tag{1.24}$$

$$\mathrm{d}\tau_{\mathrm{N}-1} \equiv \frac{\mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2 \ldots \mathrm{d}^3\boldsymbol{r}_{\mathrm{N}} \mathrm{d}^3\boldsymbol{p}_{\mathrm{N}}}{(\mathrm{N}-1)!} \tag{1.25}$$

$$A(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_{\mathrm{N}}, \boldsymbol{p}_{\mathrm{N}}) = \sum_{i=1}^{\mathrm{N}} a(\boldsymbol{r}_i, \boldsymbol{p}_i) \ , \tag{1.26}$$

$$\langle A \rangle \equiv \int A\, D_{\mathrm{N}}\, \mathrm{d}\tau_{\mathrm{N}} = \int a(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} \ . \tag{1.27}$$

$$f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2) \equiv \int D_{\mathrm{N}}\, \mathrm{d}\tau_{\mathrm{N}-2} \ , \tag{1.28}$$

$$B(\boldsymbol{r}_1, \boldsymbol{p}_1, \ldots, \boldsymbol{r}_{\mathrm{N}}, \boldsymbol{p}_{\mathrm{N}}) = \sum_{i=1}^{N} \sum_{j=1, j\neq i}^{N} b(\boldsymbol{r}_i, \boldsymbol{p}_i, \boldsymbol{r}_j, \boldsymbol{p}_j) \ , \tag{1.29}$$

$$\langle B \rangle = \int b(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, \mathrm{d}^3\boldsymbol{r}_1\, \mathrm{d}^3\boldsymbol{p}_1 \mathrm{d}^3\boldsymbol{r}_2\, \mathrm{d}^3\boldsymbol{p}_2 \ . \tag{1.30}$$

$$\mathcal{H} = \sum_{i=1}^{\mathrm{N}} \frac{\boldsymbol{p}_i^2}{2m} + \sum_{i=1}^{\mathrm{N}} V(\boldsymbol{r}_i) + \sum_{i=1}^{\mathrm{N}} \sum_{\substack{j=1 \\ j\neq i}}^{\mathrm{N}} w(\boldsymbol{r}_i, \boldsymbol{r}_j) \ . \tag{1.31}$$

$$\langle \mathcal{H} \rangle = \int h(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}, \boldsymbol{p})\, \mathrm{d}^3\boldsymbol{r}\, \mathrm{d}^3\boldsymbol{p} + \int w(\boldsymbol{r}_1, \boldsymbol{r}_2)\, f_2(\boldsymbol{r}_1, \boldsymbol{p}_1, \boldsymbol{r}_2, \boldsymbol{p}_2)\, \mathrm{d}^3\boldsymbol{r}_1\, \mathrm{d}^3\boldsymbol{p}_1\, \mathrm{d}^3\boldsymbol{r}_2\, \mathrm{d}^3\boldsymbol{p}_2 \ , \tag{1.32}$$

$$h(\boldsymbol{r}, \boldsymbol{p}) \equiv \frac{p^2}{2m} + V(\boldsymbol{r}) \ . \tag{1.33}$$

$$w(\boldsymbol{r}_i, \boldsymbol{r}_j) = -\frac{1}{2} \frac{G m^2}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \tag{1.34}$$

1.4 Boltzmann equation

$$\int \frac{\partial D_{\mathrm{N}}}{\partial t}\,\mathrm{d}\tau_{\mathrm{N}-1} = \int \sum_{i=1}^{\mathrm{N}} \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i} \cdot \frac{\partial h}{\partial \boldsymbol{r}_i} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_i} \cdot \frac{\partial h}{\partial \boldsymbol{p}_i} \right) \mathrm{d}\tau_{\mathrm{N}-1} + \int \sum_{i,j} \frac{\partial w(\boldsymbol{r}_i, \boldsymbol{r}_j)}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i}\,\mathrm{d}\tau_{\mathrm{N}-1}\ . \tag{1.35}$$

$$\int \frac{\partial D_{\mathrm{N}}}{\partial t}\,\mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial}{\partial t} \int D_{\mathrm{N}}\,\mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial f(\boldsymbol{r}, \boldsymbol{p})}{\partial t}\ . \tag{1.36}$$

$$\int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}} \cdot \frac{\partial h}{\partial \boldsymbol{r}}\,\mathrm{d}\tau_{\mathrm{N}-1} - \int \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}} \cdot \frac{\partial h}{\partial \boldsymbol{p}}\,\mathrm{d}\tau_{\mathrm{N}-1} = \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} - \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}}\ . \tag{1.37}$$

$$I_1 \equiv \int \int \left(\frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_2} \cdot \frac{\partial h}{\partial \boldsymbol{r}_2} - \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{r}_2} \cdot \frac{\partial h}{\partial \boldsymbol{p}_2} \right) \mathrm{d}^3\boldsymbol{r}_2 \mathrm{d}^3\boldsymbol{p}_2\ . \tag{1.38}$$

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \int \sum_{i,j} \frac{\partial w(\boldsymbol{r}_i, \boldsymbol{r}_j)}{\partial \boldsymbol{r}_i} \cdot \frac{\partial D_{\mathrm{N}}}{\partial \boldsymbol{p}_i}\,\mathrm{d}\tau_{\mathrm{N}-1}\ . \tag{1.39}$$

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \frac{\partial h}{\partial \boldsymbol{r}} \cdot \frac{\partial f}{\partial \boldsymbol{p}} = \left(\frac{\partial f}{\partial t} \right)_{\mathrm{coll}}\ . \tag{1.40}$$

$$f_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}') = f(\boldsymbol{r}, \boldsymbol{p})\, f(\boldsymbol{r}', \boldsymbol{p}') + g_2(\boldsymbol{r}, \boldsymbol{p}, \boldsymbol{r}', \boldsymbol{p}')\ . \tag{1.41}$$

$$\Phi(\boldsymbol{r}) \equiv \langle W \rangle(\boldsymbol{r}) = \int w(\boldsymbol{r}, \boldsymbol{r}')\, f(\boldsymbol{r}', \boldsymbol{p}')\,\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' = -\int \frac{Gm^2}{|\boldsymbol{r} - \boldsymbol{r}'|} f(\boldsymbol{r}', \boldsymbol{p}')\,\mathrm{d}^3\boldsymbol{r}'\mathrm{d}^3\boldsymbol{p}' \tag{1.42}$$

$$\frac{\partial f}{\partial t} + \frac{\partial h}{\partial \boldsymbol{p}} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \boldsymbol{\nabla} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r}) \right] \cdot \frac{\partial f}{\partial \boldsymbol{p}} = 0\ , \tag{1.43}$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}} - \tfrac{1}{m} \boldsymbol{\nabla} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r}) \right] \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0\ . \tag{1.44}$$

$$\frac{\partial f(\boldsymbol{r}, \boldsymbol{v})}{\partial t} + \boldsymbol{v} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v})}{\partial \boldsymbol{r}} + \frac{\boldsymbol{F}(\boldsymbol{r})}{m} \cdot \frac{\partial f(\boldsymbol{r}, \boldsymbol{v})}{\partial \boldsymbol{v}} = 0 \tag{1.45}$$

1.5 Fluid equations

$$\begin{pmatrix} \rho(\boldsymbol{r}; t) \\ \rho(\boldsymbol{r}; t)\, \boldsymbol{u}(\boldsymbol{r}; t) \\ \rho(\boldsymbol{r}; t)\, \mathcal{E}(\boldsymbol{r}; t) \end{pmatrix} \equiv \int \begin{pmatrix} m \\ m\, \boldsymbol{v} \\ \frac{1}{2} m\, |\boldsymbol{v} - \boldsymbol{u}|^2 \end{pmatrix} f(\boldsymbol{r}; \boldsymbol{t}, \boldsymbol{v})\,\mathrm{d}^3\boldsymbol{v}\ . \tag{1.46}$$

$$\int m \frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} + \int m\boldsymbol{v} \cdot \frac{\partial f}{\partial \boldsymbol{r}}\,\mathrm{d}^3\boldsymbol{v} - \int \boldsymbol{\nabla} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r}) \right] \cdot \frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.47}$$

$$\frac{\partial}{\partial t} \int m\, f\,\mathrm{d}^3\boldsymbol{v} + \frac{\partial}{\partial \boldsymbol{r}} \cdot \int m\, \boldsymbol{v} f\,\mathrm{d}^3\boldsymbol{v} - \boldsymbol{\nabla} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r}) \right] \cdot \int \frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.48}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\, \boldsymbol{u}) = 0\ . \tag{1.49}$$

$$\int m\, v_i \frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} + \int m\, v_i\, v_j \frac{\partial f}{\partial r_j}\,\mathrm{d}^3\boldsymbol{v} - \int v_i \frac{\partial}{\partial r_j} \left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r}) \right] \frac{\partial f}{\partial v_j}\,\mathrm{d}^3\boldsymbol{v} = 0\ . \tag{1.50}$$

$$\frac{\partial \rho\, u_i}{\partial t} + \frac{\partial}{\partial r_j}(\rho \langle v_i\, v_j \rangle) - \frac{\partial}{\partial r_j}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \int v_i\, \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\boldsymbol{v} = 0\ , \tag{1.51}$$

$$\int v_i\, \frac{\partial f}{\partial v_j}\, \mathrm{d}^3\boldsymbol{v} = -\frac{\rho}{m}\, \delta_{ij} \tag{1.52}$$

$$\langle v_i\, v_j \rangle = u_i\, u_j + \langle w_i\, w_j \rangle\ ,^1 \tag{1.53}$$

$$\frac{\partial}{\partial t}(\rho\, u_i) + \frac{\partial}{\partial r_j}(\rho\, u_i\, u_j + P\delta_{ij} - \pi_{ij}) = -\frac{\rho}{m}\frac{\partial}{\partial r_i}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] \tag{1.54}$$

$$u_i\frac{\partial \rho}{\partial t} + u_i\frac{\partial}{\partial r_j}(\rho\, u_j) + \rho\frac{\partial u_i}{\partial t} + \rho\, u_j\frac{\partial u_i}{\partial r_j} = -\frac{\rho}{m}\frac{\partial}{\partial r_i}\left[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})\right] - \frac{\partial P}{\partial r_i} + \frac{\partial \pi_{ij}}{\partial r_j}\ . \tag{1.55}$$

$$\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial r_j} = -\frac{1}{m}\frac{\partial}{\partial r_i}[V(\boldsymbol{r}) + \Phi(\boldsymbol{r})] - \frac{1}{\rho}\frac{\partial P}{\partial r_i} + \frac{1}{\rho}\frac{\partial \pi_{ij}}{\partial r_j}\ . \tag{1.56}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{f} - \frac{1}{\rho}\boldsymbol{\nabla} P + \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{\pi}\ , \tag{1.57}$$

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}, \tag{1.58}$$

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho(u^2 + \langle w^2 \rangle)\right] + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho\langle (u_i + w_i)|\boldsymbol{u} + \boldsymbol{w}|^2 \rangle\right] = -\frac{\rho}{m}\, u_i\frac{\partial}{\partial r_i}(V + \Phi)\ , \tag{1.59}$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \rho\mathcal{E}\right) + \frac{\partial}{\partial r_i}\left[\tfrac{1}{2}\rho u^2\, u_i + u_j(P\delta_{ij} - \pi_{ij}) + \rho\mathcal{E}\, u_i + F_i\right] = -\frac{\rho}{m}\, u_i\frac{\partial}{\partial r_i}(V + \Phi)\ , \tag{1.60}$$

$$\frac{\partial}{\partial t}(\tfrac{1}{2}\rho\, u^2) + \frac{\partial}{\partial r_i}(\tfrac{1}{2}\rho\, u^2\, u_i) = -\frac{\rho}{m}\, u_i\frac{\partial}{\partial r_i}(V + \Phi) - u_i\frac{\partial P}{\partial r_i} + u_i\frac{\partial \pi_{ij}}{\partial r_j}\ , \tag{1.61}$$

$$\frac{\partial}{\partial t}(\rho\,\mathcal{E}) + \frac{\partial}{\partial r_i}\left(\rho\,\mathcal{E}\, u_i\right) = -P\frac{\partial u_i}{\partial r_i} - \frac{\partial F_i}{\partial r_i} + \Psi\ , \tag{1.62}$$

$$\rho\frac{\mathrm{d}\mathcal{E}}{\mathrm{d}t} = -P\,\boldsymbol{\nabla}\cdot\boldsymbol{u} - \boldsymbol{\nabla}\cdot\boldsymbol{F} + \Psi\ . \tag{1.63}$$

$$\pi_{ij} = \mu D_{ij} = \mu\left[\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i} - \frac{3}{2}\frac{\partial u_k}{\partial r_k}\delta_{ij}\right]\ , \tag{1.64}$$

$$\boldsymbol{F} = -\kappa\,\boldsymbol{\nabla} T\ . \tag{1.65}$$

$$\triangle\Phi = 4\pi\, G\,\rho\ , \tag{1.66}$$

$$\triangle V = 4\pi\, G\,\rho_{\mathrm{ext}}\ , \tag{1.67}$$

[1] $\langle v_i v_j \rangle = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\, f v_i v_j = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\, f(u_i + w_i)(u_j + w_j) = \frac{m}{\rho}\int \mathrm{d}^3\boldsymbol{v}\ [f u_i u_j + f u_i w_j + f u_j w_i + w_i w_j]$. Considering that u_i may be taken out of the integrals and $\int w_i f \mathrm{d}^3\boldsymbol{v} = \int (v_i - u_i) f \mathrm{d}^3\boldsymbol{v} = \int v_i f \mathrm{d}^3\boldsymbol{v} - u_i \int f \mathrm{d}^3\boldsymbol{v} = \frac{\rho}{m} u_i - u_i \frac{\rho}{m} = 0$ and using the definition of the mean value, we have $\langle v_i\, v_j \rangle = u_i\, u_j + \langle w_i\, w_j \rangle$.

1.6 Thermodynamical equilibria

$$S(D) \equiv -K_{\rm B} {\rm Tr}\,(D \ln D) \;, \tag{1.68}$$

$$\eth \left\{ \frac{1}{K_{\rm B}} S(D) - \sum_{i=0}^{n} \lambda_i \left[{\rm Tr}(A_i D) - \langle A_i \rangle \right] \right\} = 0 \;, \tag{1.69}$$

$$-{\rm Tr}\left[(\ln D + \mathbb{1}) \eth D \right] - \lambda_0 {\rm Tr}(\eth D) - \sum_{i=1}^{n} \lambda_i {\rm Tr}(A_i \eth D) = 0 \,, \tag{1.70}$$

$$-{\rm Tr}\left[\ln(D) \eth D \right] - \lambda_0' {\rm Tr}(\eth D) - \sum_{i=1}^{n} \lambda_i {\rm Tr}(A_i \eth D) = 0 \,, \tag{1.71}$$

$${\rm Tr}\left[\left(-\ln D - \sum_{i=0}^{n} \lambda_i A_i \right) \eth D \right] = 0 \;. \tag{1.72}$$

$$D = \exp\left[-\sum_{i=0}^{n} \lambda_i A_i \right] . \tag{1.73}$$

$$D = {\rm e}^{-\lambda_0} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i \right] \;. \tag{1.74}$$

$${\rm e}^{-\lambda_0} {\rm Tr} \left\{ \exp\left[-\sum_{i=1}^{n} \lambda_i A_i \right] \right\} = 1 \;. \tag{1.75}$$

$$Z \equiv {\rm e}^{\lambda_0} = {\rm Tr} \left\{ \exp\left[-\sum_{i=1}^{n} \lambda_i A_i \right] \right\} \;. \tag{1.76}$$

$$D = \frac{1}{Z} \exp\left[-\sum_{i=1}^{n} \lambda_i A_i \right] \;. \tag{1.77}$$

$$\frac{\partial \ln Z}{\partial \lambda_i} = -\langle A_i \rangle \quad \text{and} \quad \frac{\partial^2 \ln Z}{\partial \lambda_i \partial \lambda_j} = \langle A_i A_j \rangle - \langle A_i \rangle \langle A_j \rangle \;. \tag{1.78}$$

$$\frac{\partial S}{\partial \langle A_i \rangle} = K_{\rm B} \lambda_i \;. \tag{1.79}$$

$$D_{\rm N} = \frac{1}{Z_{\rm M}} . \tag{1.80}$$

$$D_{\rm N} = \frac{1}{Z_{\rm C}} \exp\left[-\beta H \right] , \tag{1.81}$$

$$D = \frac{1}{Z_{\rm G}} \exp\left[-\beta H + \alpha N \right] , \tag{1.82}$$

$$D = \frac{1}{Z_{\rm G}} \exp\left[-\beta H + \alpha N \right] , \tag{1.83}$$

$$Z_\mathrm{G} = \sum_{N=0}^{\infty} e^{\alpha N} Z_\mathrm{C}(\beta, N), \tag{1.84}$$

$$H_N = \sum_i H_1(\boldsymbol{r}_i, \boldsymbol{p}_i), \tag{1.85}$$

$$H_1(\boldsymbol{r}, \boldsymbol{p}) = p^2/(2m) + V(\boldsymbol{r}) \tag{1.86}$$

$$Z_\mathrm{C} = \int\limits_{-\infty}^{+\infty} \exp\left(-\beta H_N\right) \mathrm{d}\tau_N, \tag{1.87}$$

$$\mathrm{d}\tau_N \equiv \frac{\mathrm{d}^3\boldsymbol{r}_1 \mathrm{d}^3\boldsymbol{p}_1 \ldots \mathrm{d}^3\boldsymbol{r}_N \mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N} N!}. \tag{1.88}$$

$$Z_\mathrm{C}(\beta, N) = \int \exp\left[-\beta H_N\right] \mathrm{d}\tau_N = \int\limits_{-\infty}^{+\infty} \prod_{i=1}^{N} \left(e^{-\beta p_i^2/(2m)} e^{-\beta V(\boldsymbol{r}_i)} \mathrm{d}^3\boldsymbol{r}_i \mathrm{d}^3\boldsymbol{p}_i \right) \frac{1}{\hbar^{3N} N!} =$$
$$= \frac{1}{\hbar^{3N} N!} \left[\int\limits_{-\infty}^{+\infty} e^{-\beta p^2/(2m)} e^{-\beta V(\boldsymbol{r})} \mathrm{d}^3\boldsymbol{r} \mathrm{d}^3\boldsymbol{p} \right]^N. \tag{1.89}$$

$$\Omega\Xi \equiv \int\limits_{-\infty}^{+\infty} e^{-\beta V(\boldsymbol{r})} \mathrm{d}^3\boldsymbol{r}. \tag{1.90}$$

$$Z_\mathrm{C}(\beta, N) = \frac{1}{\hbar^{3N} N!} \left(2\pi m K_\mathrm{B} T\right)^{3N/2} \Omega^N \Xi^N \equiv \frac{Z_1^N}{N!}, \tag{1.91}$$

$$Z_1 \equiv \frac{\left(2\pi m K_\mathrm{B} T\right)^{3/2}}{\hbar^{3N}} \Omega\Xi. \tag{1.92}$$

$$Z_\mathrm{G} = \sum_{N=0}^{\infty} \frac{1}{N!} \left(e^\alpha Z_1\right)^N = \exp\left[e^\alpha Z_1\right], \tag{1.93}$$

$$f(\boldsymbol{r}, \boldsymbol{p}) \equiv \sum_{N=0}^{\infty} \left[\mathcal{P}(N) \sum_{i=1}^{N} \mathcal{P}_N(i, \boldsymbol{r}, \boldsymbol{p}) \right], \tag{1.94}$$

$$D \sim \begin{bmatrix} \frac{1}{Z_\mathrm{G}} & & & \\ & \frac{1}{Z_\mathrm{G}} e^\alpha \exp(-\beta H_1) & & \\ & & \frac{1}{Z_\mathrm{G}} e^{2\alpha} \exp(-\beta[H_1(\boldsymbol{r}_1, \boldsymbol{p}_1) + H_1(\boldsymbol{r}_2, \boldsymbol{p}_2)]) & \\ & & & \ddots \end{bmatrix} \tag{1.95}$$

$$D_{N,N} = \frac{1}{Z_\mathrm{G}} e^{\alpha N} \exp\left[-\beta \sum_{i=1}^{N} H_1(\boldsymbol{r}_i, \boldsymbol{p}_i) \right]. \tag{1.96}$$

$$\mathcal{P}(N)=\int_{-\infty}^{+\infty} D_{N,N}\mathrm{d}\tau_N=\frac{e^{\alpha N}Z_{\mathrm{C}}(\beta,N)}{Z_{\mathrm{G}}(\alpha,\beta)}. \tag{1.97}$$

$$\mathcal{P}_N(1,\boldsymbol{r},\boldsymbol{p})=\int_{-\infty}^{+\infty} D_N\frac{\mathrm{d}^3\boldsymbol{r}_2\mathrm{d}^3\boldsymbol{p}_2\ldots\mathrm{d}^3\boldsymbol{r}_N\mathrm{d}^3\boldsymbol{p}_N}{\hbar^{3N}N!}=\frac{1}{N\hbar^3}\int_{-\infty}^{+\infty} D_N\mathrm{d}\tau_{N-1} \tag{1.98}$$

$$\begin{aligned}\mathcal{P}_N(1,\boldsymbol{r},\boldsymbol{p})&=\frac{1}{N\hbar^3}\int_{-\infty}^{+\infty}\frac{e^{-\beta H_N}\mathrm{d}\tau_{N-1}}{\frac{1}{N!}Z_1^N(\beta)}=\frac{1}{N\hbar^3}\int_{-\infty}^{+\infty}\frac{N(N-1)!e^{-\beta H_N}\mathrm{d}\tau_{N-1}}{Z_1^N(\beta)}=\\&=\frac{(N-1)!}{\hbar^3}e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})}\frac{1}{Z_1^N}\int e^{-\beta H_{N-1}}\mathrm{d}\tau_{N-1}=\frac{1}{\hbar^3}\frac{e^{-\beta H_1(\boldsymbol{r},\boldsymbol{p})}}{Z_1(\beta)}.\end{aligned} \tag{1.99}$$

$$\int e^{-\beta H_{N-1}}\mathrm{d}\tau_{N-1}=\frac{1}{(N-1)!}Z_1^{(N-1)}, \tag{1.100}$$

$$f(\boldsymbol{r},\boldsymbol{p})=\sum_{N=0}^{\infty}\frac{e^{\alpha N}Z_{\mathrm{C}}(\beta,N)}{Z_{\mathrm{G}}(\beta,\alpha)}N\frac{1}{\hbar^3}\frac{e^{-\beta H_1}}{Z_1}, \tag{1.101}$$

$$f(\boldsymbol{r},\boldsymbol{p})=\frac{1}{\hbar^3 Z_{\mathrm{G}}}\sum_{N=0}^{\infty}\frac{e^{\alpha N}Z_1^N}{N!}N\frac{e^{-\beta H_1}}{Z_1}=\frac{e^{\alpha}e^{-\beta H_1}}{\hbar^3}\frac{\sum_{N=1}^{\infty}\frac{e^{\alpha(N-1)}Z_1^{N-1}}{(N-1)!}}{\exp\left[e^{\alpha}Z_1\right]}, \tag{1.102}$$

$$f(\boldsymbol{r},\boldsymbol{p})=\frac{e^{\alpha-\beta H_1(\boldsymbol{r},\boldsymbol{p})}}{\hbar^3}. \tag{1.103}$$

$$\langle N\rangle=\frac{\partial\ln Z_{\mathrm{G}}}{\partial\alpha}=\frac{\partial}{\partial\alpha}e^{\alpha}Z_1=e^{\alpha}Z_1 \tag{1.104}$$

$$\begin{aligned}f'(\boldsymbol{p})&=\int_{-\infty}^{+\infty}f(\boldsymbol{r},\boldsymbol{p})\mathrm{d}^3\boldsymbol{r}=\int_{-\infty}^{+\infty}\mathrm{d}^3\boldsymbol{r}\frac{1}{\hbar^3}e^{\alpha}\exp\left\{-\beta\left[p^2/(2m)+V(\boldsymbol{r})\right]\right\}=\\&=\frac{\Omega}{\hbar^3}\exp\left[\alpha-\beta\frac{p^2}{2m}\right]=\langle N\rangle g(\boldsymbol{p}).\end{aligned} \tag{1.105}$$

$$g(\boldsymbol{p})=\frac{\Omega}{\hbar^3}e^{-\frac{\beta p^2}{2m}}\frac{1}{Z_1}=\frac{e^{-\beta p^2/2m}}{(2\pi K_{\mathrm{B}}mT)^{3/2}}=\frac{e^{-\frac{p^2}{2K_{\mathrm{B}}Tm}}}{(2\pi K_{\mathrm{B}}mT)^{3/2}}. \tag{1.106}$$

$$H=\sum_{i=1}^{\mathrm{N^e}}H_1^{\mathrm{e}}(\boldsymbol{r}_i\,\boldsymbol{p}_i)+\sum_{i=1}^{\mathrm{N^p}}H_1^{\mathrm{p}}(\boldsymbol{r}_i\,\boldsymbol{p}_i)+\sum_{i=1}^{\mathrm{N^H}}H_1^{\mathrm{H}}(\boldsymbol{r}_i\,\boldsymbol{p}_i) \tag{1.107}$$

$$H_1^a(\boldsymbol{r},\,\boldsymbol{p})=\frac{p^2}{2m^a}+V(\boldsymbol{r})+h_1^a\,,\quad a=\{\mathrm{e},\,\mathrm{p},\,\mathrm{H}\}\,. \tag{1.108}$$

$$Z_{\mathrm{G}}(\beta,\, \mu^{\mathrm{e}},\, \mu^{\mathrm{p}},\, \mu^{\mathrm{H}}) = \prod_a \left\{\mathrm{Tr}\left[\exp(\beta\mu^a N^a)\right]\ \mathrm{Tr}\left[\exp(-\beta H^a)\right]\right\}\ . \tag{1.109}$$

$$\Omega \equiv \int \exp\left[-\beta\, V(\boldsymbol{r})\right]\mathrm{d}^3\boldsymbol{r}\ , \tag{1.110}$$

$$\zeta^a(T) \equiv \mathrm{Tr}\Big\{\exp\left(-\beta h_1^a\right)\Big\} = \sum d_i^a \exp\left[-\beta\mathcal{E}_i^a\right]\,, \tag{1.111}$$

$$Z_{\mathrm{G}}(\beta,\, \mu^{\mathrm{e}},\, \mu^{\mathrm{p}},\, \mu^{\mathrm{H}}) = \prod_a \sum_{\mathrm{N}^a} \frac{1}{\mathrm{N}^a!}\left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right]^{\mathrm{N}^a} = \prod_a \exp\left[\mathrm{e}^{\beta\mu^a} Z_1^a\, \zeta^a(T)\right]\ . \tag{1.112}$$

$$\langle \mathrm{N}^a\rangle = \frac{1}{\beta}\frac{\partial \ln Z_{\mathrm{G}}}{\partial \mu^a} = \frac{(2\pi m^a K_{\mathrm{B}}T)^{3/2}}{\hbar^3}\Omega\, \zeta^a(T)\exp\left(\frac{\mu^a}{K_{\mathrm{B}}T}\right)\ . \tag{1.113}$$

$$\mu^{\mathrm{e}} + \mu^{\mathrm{p}} = \mu^{\mathrm{H}}\ . \tag{1.114}$$

$$\prod_a \left[\exp\left(\frac{\mu^a}{K_{\mathrm{B}}T}\right)\right]^{\nu^a} = 1\,,\ \text{with}\quad \nu^{\mathrm{e}} = \nu^{\mathrm{p}} = 1,\ \nu^{\mathrm{H}} = -1\ . \tag{1.115}$$

$$\prod_a \left[n^a \frac{\hbar^3}{(2\pi m^a K_{\mathrm{B}}T)^{3/2}}\frac{1}{\zeta^a(T)}\right]^{\nu^a} = 1 \tag{1.116}$$

$$\frac{n_{\mathrm{e}}}{n_{\mathrm{H}}} = \frac{4}{d_0}\,\frac{1}{n_{\mathrm{e}}}\,\frac{(2\pi m^{\mathrm{e}} K_{\mathrm{B}}T)^{3/2}}{\hbar^3}\ \exp\left(-\frac{U_{\mathrm{i}}}{K_{\mathrm{B}}T}\right)\ . \tag{1.117}$$

Chapter 2

Plasma

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho_{\mathrm{e}}(\boldsymbol{r})}{\varepsilon_0}, \tag{2.1}$$

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla\phi(\boldsymbol{r}), \tag{2.2}$$

$$\triangle\phi(\boldsymbol{r}) = -\frac{\rho_{\mathrm{e}}(\boldsymbol{r})}{\varepsilon_0}. \tag{2.3}$$

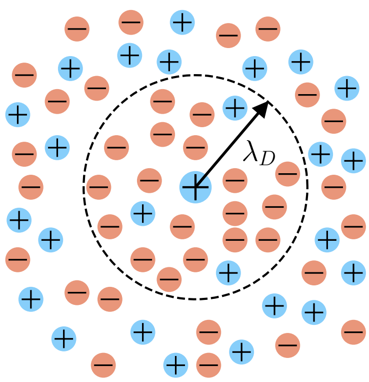
$$f_\alpha(\boldsymbol{r},\boldsymbol{v}) = A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2 + q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right], \tag{2.4}$$

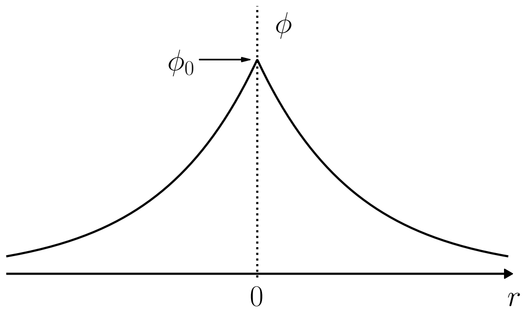
$$n_\alpha(\phi) = \int\limits_0^\infty f_\alpha(\boldsymbol{v})\mathrm{d}^3\boldsymbol{v} = \exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right]\int\limits_0^\infty A_\alpha \exp\left[-\frac{\frac{1}{2}m_\alpha v^2}{K_{\mathrm{B}}T_\alpha}\right]\mathrm{d}^3\boldsymbol{v}. \tag{2.5}$$

$$n_\alpha(\phi) = n_{\alpha,0}\exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right]. \tag{2.6}$$

$$\rho_{\mathrm{e}}(\phi) = \sum_\alpha q_\alpha n_\alpha(\phi(\boldsymbol{r})). \tag{2.7}$$

$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0}\sum_\alpha q_\alpha n_{\alpha,0}\exp\left[-\frac{q_\alpha\phi(\boldsymbol{r})}{K_{\mathrm{B}}T_\alpha}\right]. \tag{2.8}$$





$$\triangle\phi(\boldsymbol{r}) = -\frac{1}{\varepsilon_0}\left[\sum_\alpha q_\alpha n_{\alpha,0} - \sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}\phi(\boldsymbol{r})}{K_\mathrm{B}T_\alpha} + \dots\right]. \tag{2.9}$$

$$\triangle\phi(\boldsymbol{r}) = \left[\sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_\mathrm{B}T_\alpha}\right]\phi(\boldsymbol{r}). \tag{2.10}$$

$$\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left[r\phi(r)\right] = a\phi(r), \tag{2.11}$$

$$a = \sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_\mathrm{B}T_\alpha} \tag{2.12}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}\psi(r) = a\psi(r), \tag{2.13}$$

$$\psi(r) = c_1 e^{\sqrt{a}r} + c_2 e^{-\sqrt{a}r}. \tag{2.14}$$

$$\phi(r) = \frac{c_2}{r}e^{-\sqrt{a}r}. \tag{2.15}$$

$$\frac{c_2}{r} \to \frac{Q}{4\pi\varepsilon_0 r}. \tag{2.16}$$

$$\phi(r) = \frac{Q}{4\pi\varepsilon_0 r}\exp\left[-\frac{r}{\lambda_\mathrm{D}}\right], \tag{2.17}$$

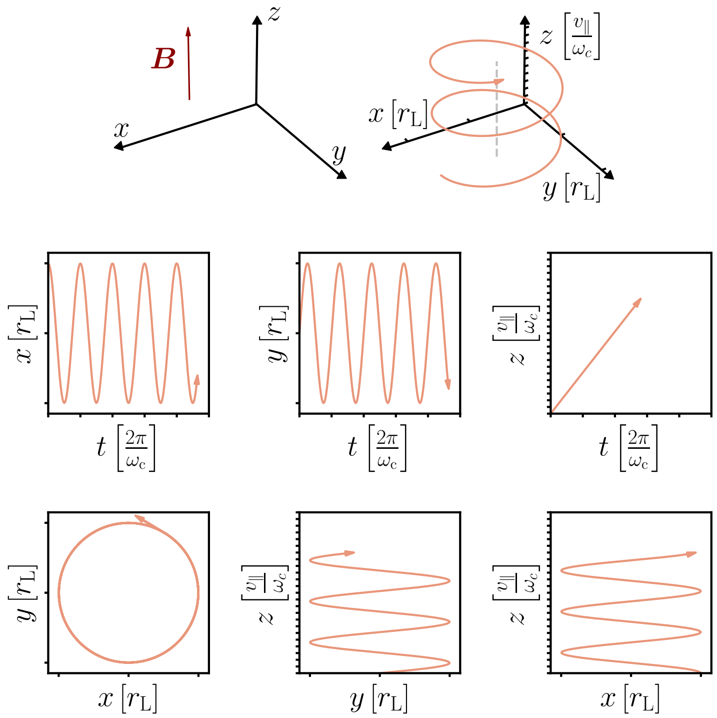
$$\lambda_\mathrm{D} \equiv \left[\sum_\alpha \frac{q_\alpha^2 n_{\alpha,0}}{\varepsilon_0 K_\mathrm{B}T_\alpha}\right]^{-1/2} \tag{2.18}$$

$$\lambda_\mathrm{D} \equiv \sqrt{\frac{\varepsilon_0 K_\mathrm{B}T_\mathrm{e}}{n_0 e^2}}, \tag{2.19}$$

$$E_\mathrm{k} = \frac{1}{2}m\bar{v}^2 = K_\mathrm{B}T \tag{2.20}$$

$$\mathrm{O}_2^+ + \mathrm{e}^- \to \mathrm{O} + \mathrm{O} + 6.6\mathrm{eV} \tag{2.21}$$

$$\mathrm{O}_2^+ + \mathrm{N}_2 \to \mathrm{NO}^+ + \mathrm{NO} + 0.87\mathrm{eV}. \tag{2.22}$$



Chapter 3

Charged particle motion in electromagnetic fields

$$m\frac{\mathrm{d}\boldsymbol{v}(\boldsymbol{r},t)}{\mathrm{d}t} = q\left[\boldsymbol{E}(\boldsymbol{r},t) + \boldsymbol{v}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t)\right], \tag{3.1}$$

3.1 Homogeneous magnetic field

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right], \tag{3.2}$$

$$m\frac{\mathrm{d}v_x}{\mathrm{d}t} = qBv_y, \tag{3.3}$$

$$m\frac{\mathrm{d}v_y}{\mathrm{d}t} = -qBv_x, \tag{3.4}$$

$$m\frac{\mathrm{d}v_z}{\mathrm{d}t} = 0. \tag{3.5}$$

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = \frac{q}{m}B\frac{\mathrm{d}v_y}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\frac{q}{m}B\frac{\mathrm{d}v_x}{\mathrm{d}t}. \tag{3.6}$$

$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_x \quad \text{and} \quad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\left(\frac{q}{m}B\right)^2 v_y, \tag{3.7}$$

$$\omega_\mathrm{c} \equiv \frac{qB\,\mathrm{sgn}\,q}{m} = \frac{|q|B}{m} \tag{3.8}$$

$$v_{x,y} = v_\perp \exp\left(\mathrm{i}\omega_\mathrm{c}t + \mathrm{i}\delta_{x,y}\right), \tag{3.9}$$

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c}t, \tag{3.10}$$

$$v_y = \frac{m}{qB}\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{m}{qB}v_\perp \mathrm{i}\omega_\mathrm{c}\exp \mathrm{i}\omega_\mathrm{c}t = \frac{m}{qB}v_\perp \mathrm{i}\frac{qB\,\mathrm{sgn}\,q}{m}\exp \mathrm{i}\omega_\mathrm{c}t = v_\perp \mathrm{i}\,\mathrm{sgn}\,q\exp \mathrm{i}\omega_\mathrm{c}t. \tag{3.11}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \Re[v_x] = \Re[v_\perp \exp \mathrm{i}\omega_\mathrm{c}t] \to x - x_0 = \Re\left[-\mathrm{i}\frac{v_\perp}{\omega_\mathrm{c}}\exp \mathrm{i}\omega_\mathrm{c}t\right] = \frac{v_\perp}{\omega_\mathrm{c}}\sin\omega_\mathrm{c}t \tag{3.12}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \Re[v_y] = \Re[v_\perp \mathrm{i}\exp \mathrm{i}\omega_\mathrm{c}t] \to y - y_0 = \Re\left[\frac{v_\perp}{\omega_\mathrm{c}}\mathrm{sgn}\,q\exp \mathrm{i}\omega_\mathrm{c}t\right] = \frac{v_\perp}{\omega_\mathrm{c}}\mathrm{sgn}\,q\cos\omega_\mathrm{c}t \tag{3.13}$$

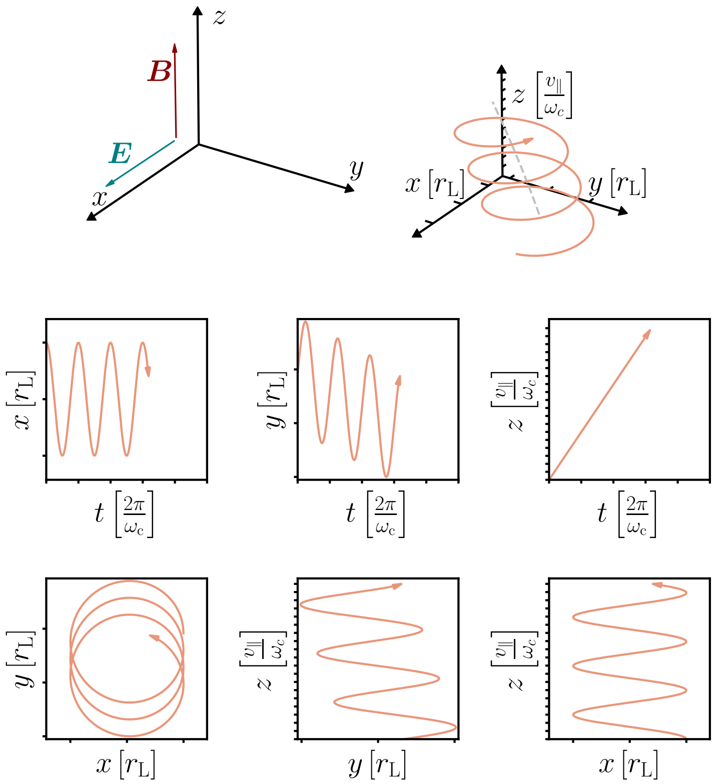
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \Re[v_z] = 0 \to z - z_0 = v_\parallel t. \tag{3.14}$$

$$r_\mathrm{L} = \frac{v_\perp m}{qB\,\mathrm{sgn}\,q} = \frac{v_\perp m}{|q|B} \tag{3.15}$$

3.2 Homogeneous electric field

$$m\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = q\left[\begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} + \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}\right]. \tag{3.16}$$

$$\frac{\mathrm{d}v_z}{\mathrm{d}t} = \frac{q}{m}E_z \to z - z_0 = \frac{q}{2m}E_z t^2 + v_{z,0}t. \tag{3.17}$$



$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y, \tag{3.18}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\omega_\mathrm{c}\,\mathrm{sgn}\,q v_x. \tag{3.19}$$

$$\frac{\mathrm{d}^2v_x}{\mathrm{d}t^2} = \omega_\mathrm{c}\,\mathrm{sgn}\,q\frac{\mathrm{d}v_y}{\mathrm{d}t} = -\,\mathrm{sgn}^{\,2} q\omega_\mathrm{c}^2 v_x = -\omega_\mathrm{c}^2 v_x, \tag{3.20}$$

$$\frac{\mathrm{d}^2v_y}{\mathrm{d}t^2} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\frac{\mathrm{d}v_x}{\mathrm{d}t} = -\,\mathrm{sgn}\,q\omega_\mathrm{c}\left(\frac{q}{m}E_x + \omega_\mathrm{c}\,\mathrm{sgn}\,q v_y\right) = -\omega_\mathrm{c}^2\left(\frac{E_x}{B} + v_y\right). \tag{3.21}$$

$$v_x = v_\perp \exp \mathrm{i}\omega_\mathrm{c} t. \tag{3.22}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_\mathrm{c}^2\left(v_y + \frac{E_x}{B}\right), \tag{3.23}$$

$$v_y = v_\perp \mathrm{i}\,\mathrm{sgn}\,q \exp \mathrm{i}\omega_\mathrm{c} t - \frac{E_x}{B}. \tag{3.24}$$

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) \tag{3.25}$$

$$m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = m\frac{\mathrm{d}\boldsymbol{v}'}{\mathrm{d}t} = q\left(\boldsymbol{E} + \boldsymbol{v}' \times \boldsymbol{B} + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\times\boldsymbol{B}\right) =$$
$$= q\left[\cancel{\boldsymbol{E}} + \boldsymbol{v}' \times \boldsymbol{B} - \frac{1}{\cancel{B^2}}\cancel{\boldsymbol{E}B^2} + \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}\right] =$$
$$= q\left[\boldsymbol{v}' \times \boldsymbol{B} + \boldsymbol{E}_\parallel\right], \tag{3.26}$$

$$\boldsymbol{E}_\parallel \equiv \frac{\boldsymbol{B}(\boldsymbol{E}\cdot\boldsymbol{B})}{B^2}. \tag{3.27}$$

$$\boldsymbol{v} = \boldsymbol{v}' + \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}. \tag{3.28}$$

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}, \tag{3.29}$$

$$\boldsymbol{F}_E = q\boldsymbol{E}. \tag{3.30}$$

$$\boldsymbol{v}_E = \frac{1}{q}\frac{\boldsymbol{F}_E\times\boldsymbol{B}}{B^2}. \tag{3.31}$$

$$\boldsymbol{v}_{\mathrm{gs}} = \frac{1}{q}\frac{\boldsymbol{F}\times\boldsymbol{B}}{B^2}. \tag{3.32}$$

$$\boldsymbol{v}_g = \frac{m}{q}\frac{\boldsymbol{g}\times\boldsymbol{B}}{B^2}, \tag{3.33}$$

$$\boldsymbol{v}_{\mathrm{c}} = \frac{mv_\parallel^2}{qR^2B^2}\boldsymbol{R}\times\boldsymbol{B} \tag{3.34}$$

3.3 Inhomogeneous magnetic field

$$L \sim \frac{|\boldsymbol{B}|}{|\nabla B|}. \tag{3.35}$$

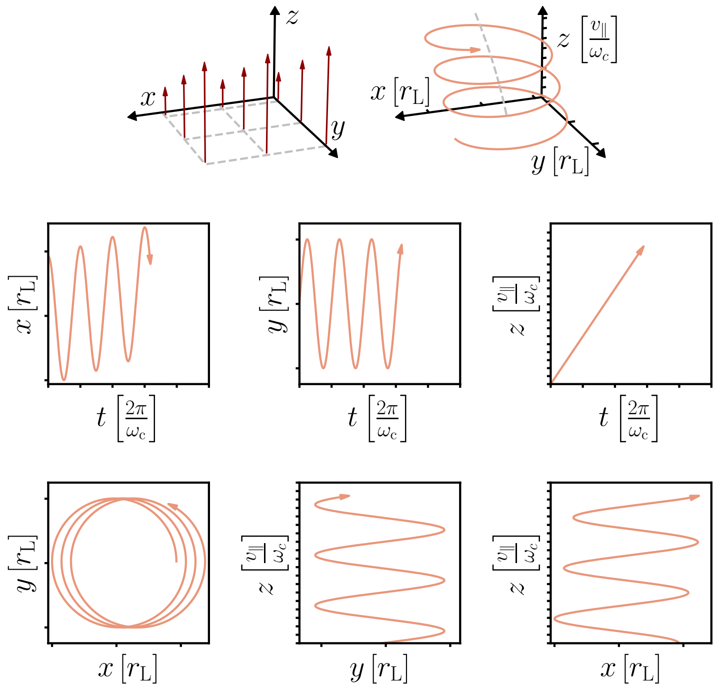
$$\boldsymbol{B} = \boldsymbol{B}_0 + (\boldsymbol{r}\cdot\nabla)\boldsymbol{B} \tag{3.36}$$

$$B_z = B_0 + y\frac{\partial B_z}{\partial y}. \tag{3.37}$$

$$F_y = -qv_xB_z = -qv_x\left(B_0 + y\frac{\partial B_z}{\partial y}\right). \tag{3.38}$$

$$F_y = -qv_\perp\cos\omega_{\mathrm{c}}t\left(B_0 + r_{\mathrm{L}}\ \mathrm{sgn}\, q\cos\omega_{\mathrm{c}}t\frac{\partial B_z}{\partial y}\right). \tag{3.39}$$

$$\langle F_y\rangle = \frac{1}{2\pi}\int\limits_0^{2\pi} F_y\mathrm{d}(\omega_{\mathrm{c}}t) = -\frac{1}{2\pi}\int\limits_0^{2\pi} qv_\perp\cos\omega_{\mathrm{c}}t\left(B_0 + r_{\mathrm{L}}\ \mathrm{sgn}\, q\cos\omega_{\mathrm{c}}t\frac{\partial B_z}{\partial y}\right)\mathrm{d}(\omega_{\mathrm{c}}t) =$$
$$= -qv_\perp B_0\frac{1}{2\pi}\int\limits_0^{2\pi}\cos\omega_{\mathrm{c}}t\mathrm{d}(\omega_{\mathrm{c}}t) - qv_\perp r_{\mathrm{L}}\ \mathrm{sgn}\, q\frac{\partial B_z}{\partial y}\frac{1}{2\pi}\int\limits_0^{2\pi}\cos^2\omega_{\mathrm{c}}t\,\mathrm{d}(\omega_{\mathrm{c}}t) =$$
$$= 0 - \frac{1}{2}v_\perp r_{\mathrm{L}}\, q\,\mathrm{sgn}\, q\frac{\partial B_z}{\partial y}. \tag{3.40}$$



$$\langle \boldsymbol{F} \rangle = -\frac{1}{2} v_\perp r_\mathrm{L}\, q \operatorname{sgn} q \nabla |\boldsymbol{B}|, \tag{3.41}$$

$$\boldsymbol{v}_{\nabla B} = \frac{1}{2} \operatorname{sgn} q v_\perp r_\mathrm{L} \frac{\boldsymbol{B} \times \nabla |\boldsymbol{B}|}{B^2}. \tag{3.42}$$

$$\Delta = \tau |\boldsymbol{v}_{\nabla B}| = \frac{2\pi}{\omega_\mathrm{c}} \frac{1}{2} v_\perp r_\mathrm{L} \frac{|\nabla B|}{|B|} = \pi r_\mathrm{L} \frac{r_\mathrm{L}}{L} \ll r_\mathrm{L} \tag{3.43}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} = 0, \tag{3.44}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{1}{R} \frac{\partial}{\partial R} (R B_\vartheta) = 0 \tag{3.45}$$

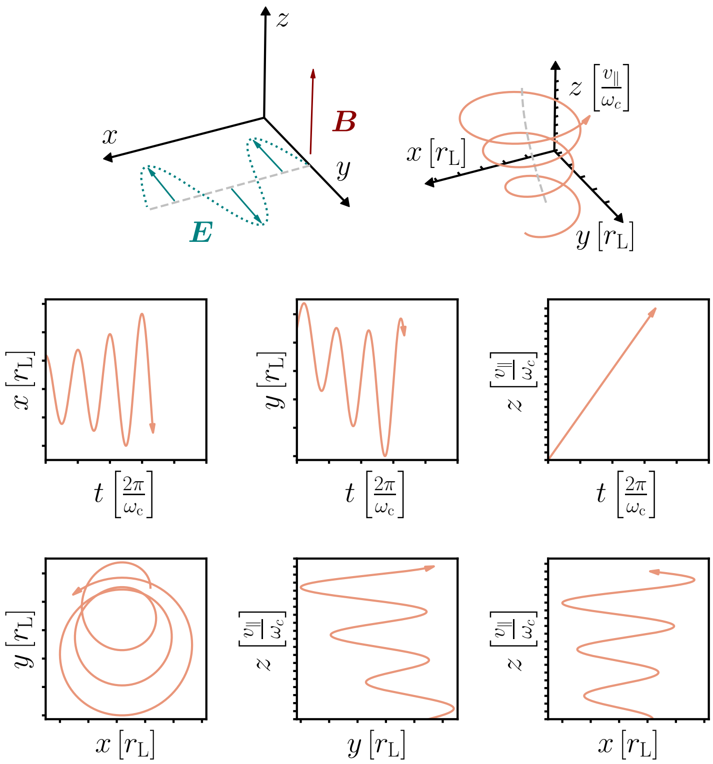
$$B_\vartheta = \frac{\mathcal{C}}{R}, \tag{3.46}$$

$$\nabla B = \frac{\partial}{\partial R} \frac{\mathcal{C}}{R} = -\mathcal{C} \frac{1}{R^2} \frac{\boldsymbol{R}}{R} \tag{3.47}$$

$$\nabla B = -\frac{|B|}{R^2} \boldsymbol{R}. \tag{3.48}$$

$$\boldsymbol{v}_{\nabla B} = -\frac{1}{2} \operatorname{sgn} q \frac{v_\perp r_\mathrm{L}}{B^2} \boldsymbol{B} \times |\boldsymbol{B}| \frac{\boldsymbol{R}}{R^2} = \frac{1}{2} \frac{m v_\perp^2}{q B^2 R^2} \boldsymbol{R} \times \boldsymbol{B}. \tag{3.49}$$

$$\boldsymbol{v}_\mathrm{gc} = \boldsymbol{v}_\mathrm{c} + \boldsymbol{v}_{\nabla B} = \frac{m}{q} \frac{\boldsymbol{R} \times \boldsymbol{B}}{R^2 B^2} \left(v_\parallel^2 + \frac{1}{2} v_\perp^2 \right). \tag{3.50}$$



3.4 Inhomogeneous electric field

$$\boldsymbol{E} = E_0 \cos ky \boldsymbol{e}_x, \tag{3.51}$$

$$\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{qB}{m}v_y + \frac{q}{m}E_x(y) \quad \text{and} \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = -\frac{qB}{m}v_x. \tag{3.52}$$

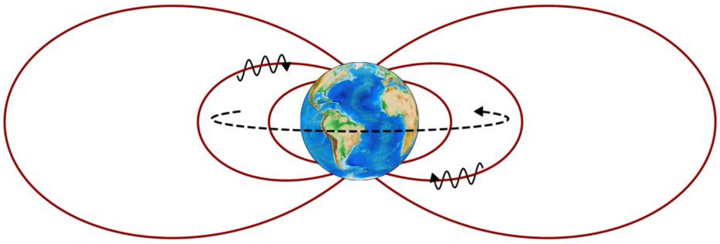
$$\frac{\mathrm{d}^2 v_x}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_x + \omega_\mathrm{c}\frac{\operatorname{sgn} q}{B}\frac{\mathrm{d}E_x}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2\frac{E_x(y)}{B}. \tag{3.53}$$

$$\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2} = -\omega_\mathrm{c}^2 v_y - \omega_\mathrm{c}^2\frac{E_0}{B}\cos\left[k(y_0 + r_\mathrm{L}\ \operatorname{sgn} q \cos\omega_\mathrm{c} t)\right], \tag{3.54}$$

$$\left\langle\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2}\right\rangle = -\omega_\mathrm{c}^2\langle v_y\rangle - \omega_\mathrm{c}^2\frac{E_0}{B}\langle\cos\left[k(y_0 + r_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t)\right]\rangle. \tag{3.55}$$

$$\left\langle\frac{\mathrm{d}^2 v_y}{\mathrm{d}t^2}\right\rangle = 0. \tag{3.56}$$

$$\begin{aligned}
\cos\left[k(y_0\right. &\left. + r_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t)\right] = \\
&= \cos ky_0 \cos\left(kr_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t\right) - \sin ky_0 \sin\left(kr_\mathrm{L}\ \operatorname{sgn} q\cos\omega_\mathrm{c} t\right) = \\
&= \cos ky_0 \cos\left(kr_\mathrm{L}\ \cos\omega_\mathrm{c} t\right) - \operatorname{sgn} q \sin ky_0 \sin\left(kr_\mathrm{L}\ \cos\omega_\mathrm{c} t\right) \\
&= \cos ky_0\left[1 - \frac{1}{2}k^2 {r_\mathrm{L}}^2\cos^2\omega_\mathrm{c} t\right] - \operatorname{sgn} q\sin ky_0 (kr_\mathrm{L}\ \cos\omega_\mathrm{c} t)
\end{aligned} \tag{3.57}$$



$$\langle v_y \rangle = -\frac{E_0}{B}\cos ky_0 \left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right) = -\frac{E_x(y)}{B}\left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right). \tag{3.58}$$

$$\boldsymbol{v}_E = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\left(1 - \frac{1}{4}k^2 {r_\mathrm{L}}^2\right) \tag{3.59}$$

$$\boldsymbol{v}_E = -\frac{\boldsymbol{B}\times}{B^2}\left(1 + \frac{1}{4}{r_\mathrm{L}}^2 \triangle\right)\boldsymbol{E}. \tag{3.60}$$

3.5 Drift currents

$$\boldsymbol{j}_\mathrm{drift} = \sum_k n_k q_k \boldsymbol{v}_{\mathrm{drift},k} = \frac{\sum_k n_k \boldsymbol{F}_k}{B^2}\times\boldsymbol{B}. \tag{3.61}$$

$$\boldsymbol{j}_{\nabla B} = \frac{1}{2}\sum_k \frac{m_k n_k v_{\perp,k}^2}{B^2}(\boldsymbol{B}\times\nabla B). \tag{3.62}$$

3.6 Guiding centre motion

$$\frac{r_\mathrm{L}}{L} = \frac{v_\perp}{\omega_\mathrm{c} L} \ll 1 \quad \text{and} \quad \frac{1}{\omega_\mathrm{c}\tau} \ll 1, \tag{3.63}$$

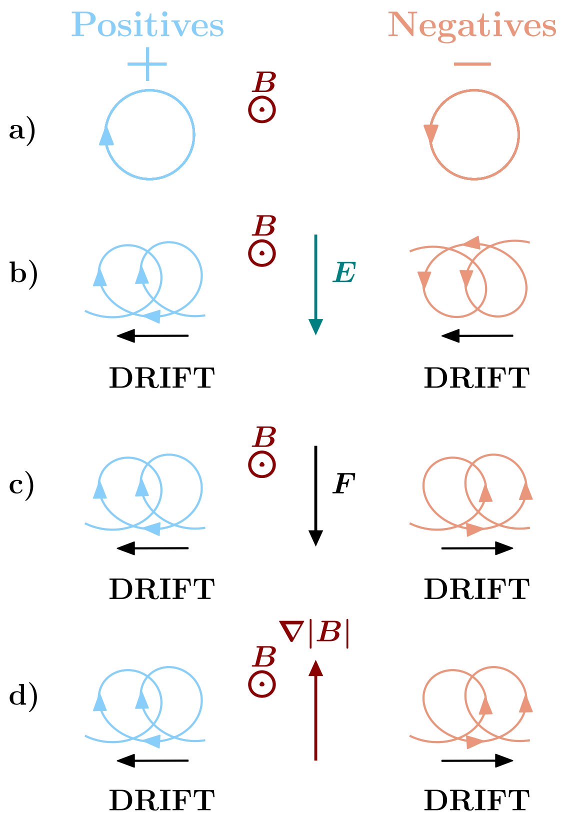
$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{v}\,, \tag{3.64}$$

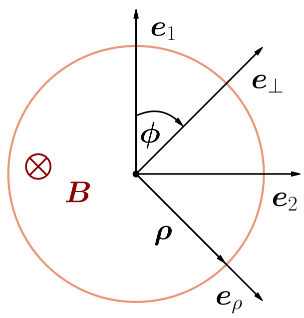
$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right). \tag{3.65}$$

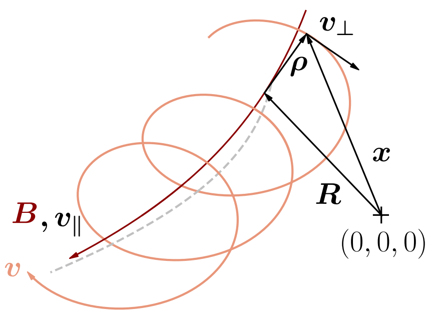
$$\boldsymbol{R}(t) = \boldsymbol{x}(t) - \boldsymbol{\rho}(t), \tag{3.66}$$

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_\mathrm{E}. \tag{3.67}$$

$$\boldsymbol{\rho} \equiv \frac{\epsilon}{B}\boldsymbol{b}\times\boldsymbol{u}, \tag{3.68}$$







$$\omega_c(t) = \frac{B(\boldsymbol{x},t)}{\epsilon} \tag{3.69}$$

$$\boldsymbol{u} = v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp\ , \quad \text{where} \quad \boldsymbol{e}_\perp \equiv \boldsymbol{e}_1 \cos\phi + \boldsymbol{e}_2 \sin\phi\ , \quad \text{and} \quad \phi = \phi_0 - \omega_c t. \tag{3.70}$$

$$\boldsymbol{e}_\rho \equiv \boldsymbol{e}_2 \cos\phi - \boldsymbol{e}_1 \sin\phi \tag{3.71}$$

$$\boldsymbol{x} = \boldsymbol{R} + \boldsymbol{\rho}, \tag{3.72}$$

$$\boldsymbol{\rho} = -\frac{m}{qB^2}\boldsymbol{u} \times \boldsymbol{B} = \rho \,\mathrm{sgn}\, q\, \boldsymbol{e}_\rho, \tag{3.73}$$

$$\boldsymbol{u}_\perp = u_\perp \boldsymbol{e}_\perp \quad \text{and} \tag{3.74}$$

$$\boldsymbol{v} = v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}. \tag{3.75}$$

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} &= \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{\rho}}{\mathrm{d}t} = \boldsymbol{v} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{u}\right) = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = \\
&= v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) - \frac{\epsilon}{B}\boldsymbol{b} \times \left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = \\
&= v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t},
\end{aligned} \tag{3.76}$$

$$\begin{aligned}
-\frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} &= -\frac{\epsilon}{B}\boldsymbol{b} \times \left[\frac{1}{\epsilon}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right)\right] = \\
&= -\frac{1}{B}\boldsymbol{b} \times \boldsymbol{E} - \frac{1}{B}(\boldsymbol{v}\,\boldsymbol{b} \cdot \boldsymbol{B} - \boldsymbol{B}\,\boldsymbol{b} \cdot \boldsymbol{v}) = -\frac{1}{B}\boldsymbol{b} \times \boldsymbol{E} - \boldsymbol{v}\frac{\boldsymbol{B} \cdot \boldsymbol{B}}{B^2} + v_\parallel \frac{\boldsymbol{B}}{B} = \\
&= \boldsymbol{v}_\mathrm{E} - \boldsymbol{v} + v_\parallel \boldsymbol{b} = -u_\perp \boldsymbol{e}_\perp.
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
\frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} &= \frac{1}{\epsilon}\left[\boldsymbol{E} + (v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}) \times \boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[\boldsymbol{E} + v_\parallel \boldsymbol{b} \times \boldsymbol{B} + u_\perp \boldsymbol{e}_\perp \times \boldsymbol{B} + \boldsymbol{v}_\mathrm{E} \times \boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[\cancel{\boldsymbol{E}} + u_\perp \boldsymbol{e}_\perp \times \boldsymbol{B} - \cancel{\frac{E}{B^2}\boldsymbol{B} \cdot \boldsymbol{B}} + \frac{\boldsymbol{E} \cdot \boldsymbol{B}}{B^2}\boldsymbol{B}\right] = \\
&= \frac{1}{\epsilon}\left[E_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp \times \boldsymbol{B}\right],
\end{aligned} \tag{3.78}$$

$$\frac{1}{\epsilon}\left[E_\parallel \boldsymbol{b} \cdot \boldsymbol{e}_\perp + u_\perp (\boldsymbol{e}_\perp \times \boldsymbol{B}) \cdot \boldsymbol{e}_\perp\right] = \left[\frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right] \cdot \boldsymbol{e}_\perp. \tag{3.79}$$

$$0 = \boldsymbol{e}_\perp \cdot v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \boldsymbol{e}_\perp \cdot \boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + \boldsymbol{e}_\perp \cdot \boldsymbol{e}_\perp \frac{\mathrm{d}u_\perp}{\mathrm{d}t} + u_\perp \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} + \boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}. \tag{3.80}$$

$$\boldsymbol{e}_\perp \cdot \frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{e}_\perp \cdot \boldsymbol{e}_\perp) = 0, \tag{3.81}$$

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right). \tag{3.82}$$

$$\frac{1}{\epsilon}\left[E_\parallel \boldsymbol{b} \cdot \boldsymbol{b} + u_\perp (\boldsymbol{e}_\perp \times \boldsymbol{B}) \cdot \boldsymbol{b}\right] = \left[\frac{\mathrm{d}(v_\parallel \boldsymbol{b})}{\mathrm{d}t} + \frac{\mathrm{d}(u_\perp \boldsymbol{e}_\perp)}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right] \cdot \boldsymbol{b}. \tag{3.83}$$

$$\frac{E_\parallel}{\epsilon} = \boldsymbol{b}\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} + v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}\cdot\boldsymbol{b} + \boldsymbol{b}\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t} + u_\perp\boldsymbol{b}\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}\cdot\boldsymbol{b}}{\mathrm{d}t} - \boldsymbol{v}_\mathrm{E}\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.84}$$

$$u_\perp\boldsymbol{b}\cdot\frac{\mathrm{d}\boldsymbol{e}_\perp}{\mathrm{d}t} = u_\perp\frac{\mathrm{d}\boldsymbol{e}_\perp\cdot\boldsymbol{b}}{\mathrm{d}t} - u_\perp\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} = -u_\perp\boldsymbol{e}_\perp\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.85}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + (u_\perp\boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E})\cdot\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}. \tag{3.86}$$

$$\frac{1}{\epsilon}\left[E_\parallel\overset{1}{\boldsymbol{b}}\cdot\boldsymbol{e}_\rho + u_\perp(\boldsymbol{e}_\perp\overset{2}{\times}\boldsymbol{B})\cdot\boldsymbol{e}_\rho\right] = \boldsymbol{e}_\rho\cdot\overset{3}{\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t}} + \boldsymbol{e}_\rho\cdot\overset{4}{\frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t}} + \boldsymbol{e}_\rho\cdot\overset{5}{\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}}. \tag{3.87}$$

$$\boldsymbol{b}\perp\boldsymbol{e}_\rho \Rightarrow \boldsymbol{b}\cdot\boldsymbol{e}_\rho = 0 \tag{3.88}$$

$$\begin{aligned} u_\perp(\boldsymbol{e}_\perp\times\boldsymbol{B})\cdot\boldsymbol{e}_\rho &= u_\perp(\boldsymbol{e}_\rho\times\boldsymbol{e}_\perp)\cdot\boldsymbol{B} = u_\perp\left[\begin{pmatrix}-\sin\phi\\ \cos\phi\\ 0\end{pmatrix}\times\begin{pmatrix}\cos\phi\\ \sin\phi\\ 0\end{pmatrix}\right]\cdot\boldsymbol{B} = \\ &= u_\perp\begin{pmatrix}0\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}0\\0\\B\end{pmatrix} = -u_\perp B \end{aligned} \tag{3.89}$$

$$\boldsymbol{e}_\rho\cdot\frac{\mathrm{d}(v_\parallel\boldsymbol{b})}{\mathrm{d}t} = \boldsymbol{e}_\rho\cdot v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho\cdot\boldsymbol{b}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}} \tag{3.90}$$

$$\begin{aligned} \boldsymbol{e}_\rho\cdot\frac{\mathrm{d}(u_\perp\boldsymbol{e}_\perp)}{\mathrm{d}t} &= u_\perp(\boldsymbol{e}_2\cos\phi - \boldsymbol{e}_1\sin\phi)\cdot\frac{\mathrm{d}(\boldsymbol{e}_1\cos\phi + \boldsymbol{e}_2\sin\phi)}{\mathrm{d}t} + \cancel{\boldsymbol{e}_\rho\cdot\boldsymbol{e}_\perp\frac{\mathrm{d}u_\perp}{\mathrm{d}t}} = \\ &= u_\perp\left[(\boldsymbol{e}_2\cos\phi - \boldsymbol{e}_1\sin\phi)\cdot(\cos\phi\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \boldsymbol{e}_1\sin\phi\frac{\mathrm{d}\phi}{\mathrm{d}t} + \sin\phi\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2\cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}t})\right] = \\ &= u_\perp\left[\cos^2\phi\,\boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \cancel{\sin\phi\cos\phi\,\boldsymbol{e}_1\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\sin\phi\cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}t}}\boldsymbol{e}_1\cdot\boldsymbol{e}_2 + \sin^2\phi\frac{\mathrm{d}\phi}{\mathrm{d}t}\boldsymbol{e}_1\cdot\boldsymbol{e}_1 + \right. \\ &\left. + \cancel{\sin\phi\cos\phi\,\boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t}} - \sin^2\phi\,\boldsymbol{e}_1\cdot\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \cos^2\frac{\mathrm{d}\phi}{\mathrm{d}t}\boldsymbol{e}_2\cdot\boldsymbol{e}_2 - \cancel{\sin\phi\cos\phi\frac{\mathrm{d}\phi}{\mathrm{d}t}}\boldsymbol{e}_1\cdot\boldsymbol{e}_2\right] = \\ &= u_\perp\left[(\sin^2\phi + \cos^2\phi)\left(\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right) - \sin^2\phi\left(\boldsymbol{e}_1\cdot\frac{\mathrm{d}\boldsymbol{e}_2}{\mathrm{d}t} + \boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}\right)\right] = \\ &= u_\perp\left[\frac{\mathrm{d}\phi}{\mathrm{d}t} + \boldsymbol{e}_2\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \sin^2\phi\left(\cancel{\frac{\mathrm{d}(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)}{\mathrm{d}t}} - \cancel{\boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}} + \cancel{\boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t}}\right)\right], \end{aligned} \tag{3.91}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2\cdot\frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho\cdot\left(v_\parallel\frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right). \tag{3.92}$$

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \epsilon \boldsymbol{u} \times \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\boldsymbol{b}}{B}\right) + \frac{\epsilon}{B}\boldsymbol{b} \times \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}, \tag{3.93}$$

$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = -\boldsymbol{e}_\perp \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right), \tag{3.94}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{E_\parallel}{\epsilon} + \left(u_\perp \boldsymbol{e}_\perp + \boldsymbol{v}_\mathrm{E}\right) \cdot \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t}, \tag{3.95}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right). \tag{3.96}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp\right) \cdot \boldsymbol{\nabla} \tag{3.97}$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp\right) \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp\right) \cdot \boldsymbol{\nabla}\right] \boldsymbol{b} + \\ \epsilon\left(v_\parallel \boldsymbol{b} + u_\perp \boldsymbol{e}_\perp\right) \times \boldsymbol{b}\left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp\right) \cdot \boldsymbol{\nabla}\right]\left(\frac{1}{B}\right) + \\ \frac{\epsilon}{B}\boldsymbol{b} \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) \cdot \boldsymbol{\nabla}\right] \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}\right) \boldsymbol{v}_\mathrm{E}\,.\end{aligned} \tag{3.98}$$

$$\frac{\epsilon}{B}\boldsymbol{b} \times \boldsymbol{b}\left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) \cdot \boldsymbol{\nabla}\right] v_\parallel = 0 \tag{3.99}$$

$$\begin{aligned}\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b} \times D_t\left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) + \frac{\epsilon}{B} v_\parallel \boldsymbol{b} \times \left[u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}\right] \boldsymbol{b} + \\ \frac{\epsilon u_\perp}{B}\boldsymbol{e}_\perp \times \left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp\right) \cdot \boldsymbol{\nabla}\right] \boldsymbol{b} - \\ \frac{\epsilon u_\perp}{B^2}\boldsymbol{e}_\perp \times \boldsymbol{b}\left[\frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + u_\perp \boldsymbol{e}_\perp\right) \cdot \boldsymbol{\nabla}\right] B + \frac{\epsilon}{B}\boldsymbol{b} \times \left(u_\perp \boldsymbol{e}_\perp \cdot \boldsymbol{\nabla}\right) \boldsymbol{v}_\mathrm{E}\,,\end{aligned} \tag{3.100}$$

$$D_t \equiv \frac{\partial}{\partial t} + \left(v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}\right) \cdot \boldsymbol{\nabla}\,. \tag{3.101}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = f_z(z) = \langle f_z \rangle + \tilde{f}_z, \tag{3.102}$$

$$\bar{z} = z + \frac{\epsilon}{B}\int_0^\phi \tilde{f}_z(\phi')\mathrm{d}\phi'. \tag{3.103}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot \frac{\mathrm{d}\boldsymbol{e}_1}{\mathrm{d}t} - \frac{1}{u_\perp}\boldsymbol{e}_\rho \cdot \left(v_\parallel \frac{\mathrm{d}\boldsymbol{b}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t}\right) = -\frac{B}{\epsilon} + \mathcal{O}(\epsilon^0). \tag{3.104}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} \cdot \frac{\partial}{\partial \boldsymbol{R}} + \frac{\mathrm{d}v_\parallel}{\mathrm{d}t}\frac{\partial}{\partial v_\parallel} + \frac{\mathrm{d}u_\perp}{\mathrm{d}t}\frac{\partial}{\partial u_\perp} + \frac{\mathrm{d}\phi}{\mathrm{d}t}\frac{\partial}{\partial \phi}. \tag{3.105}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = -\frac{B}{\epsilon}\frac{\partial}{\partial \phi} + \mathcal{O}(\epsilon^0). \tag{3.106}$$

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \frac{\mathrm{d}z}{\mathrm{d}t} - \frac{B}{\epsilon}\frac{\partial}{\partial\phi}\frac{\epsilon}{B}\int_\phi \tilde{f}_z(\phi')\mathrm{d}\phi' =$$
$$= \frac{\mathrm{d}z}{\mathrm{d}t} - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \tilde{f}_z - \tilde{f}_z + \mathcal{O}(\epsilon) = \langle f_z(z)\rangle + \mathcal{O}(\epsilon). \quad (3.107)$$

$$\boldsymbol{b}(\boldsymbol{x},t) = \boldsymbol{b}(\boldsymbol{R},t) + \boldsymbol{\rho}\cdot\nabla\boldsymbol{b}(\boldsymbol{R},t) + \dots = \boldsymbol{b}(\boldsymbol{R},t) + \epsilon\frac{u_\perp}{B}\boldsymbol{e}_\rho\cdot\nabla\boldsymbol{b}(\boldsymbol{R},t) + \dots \quad (3.108)$$

$$\langle\boldsymbol{b}(\boldsymbol{x},t)\rangle = \boldsymbol{b}(\boldsymbol{R},t) + \mathcal{O}(\epsilon). \quad (3.109)$$

$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}t} = \langle f_z(\bar{z})\rangle + \mathcal{O}(\epsilon), \quad \text{Q.E.D.} \quad (3.110)$$

$$\left\langle\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}\right\rangle = \left\langle v_\parallel\boldsymbol{b} + \boldsymbol{v}_\mathrm{E} + \frac{\epsilon}{B}\boldsymbol{b}\times D_t\left(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E}\right) + \frac{\epsilon u_\perp^2}{B}\boldsymbol{e}_\perp\times(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,\boldsymbol{b} - \frac{\epsilon u_\perp^2}{B^2}\boldsymbol{e}_\perp\times\boldsymbol{b}\,(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,B\right\rangle . \quad (3.111)$$

$$(\boldsymbol{e}_\perp\times\boldsymbol{b})\,(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,B = (\boldsymbol{e}_1\sin\phi - \boldsymbol{e}_2\cos\phi)\,(\cos\phi\nabla_1 + \sin\phi\nabla_2)\,B \quad (3.112)$$

$$\langle(\boldsymbol{e}_\perp\times\boldsymbol{b})\,(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,B\rangle = \tfrac{1}{2}\left(\boldsymbol{e}_1\nabla_2 B - \boldsymbol{e}_2\nabla_1 B\right) = -\tfrac{1}{2}\left(\boldsymbol{b}\times\boldsymbol{\nabla}B\right) . \quad (3.113)$$

$$\langle\boldsymbol{e}_\perp\times(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,\boldsymbol{b}\rangle = \tfrac{1}{2}\left(\boldsymbol{e}_1\nabla_2 - \boldsymbol{e}_2\nabla_1\right)b_3 + \tfrac{1}{2}\boldsymbol{e}_3\left(\nabla_1 b_2 - \nabla_2 b_1\right) . \quad (3.114)$$

$$\boldsymbol{b}(\boldsymbol{R}+\boldsymbol{\delta R}) = \boldsymbol{b}(\boldsymbol{R}) + \boldsymbol{\delta R}\cdot\boldsymbol{\nabla}\boldsymbol{b}(\boldsymbol{R}) + \mathcal{O}(\delta R^2) . \quad (3.115)$$

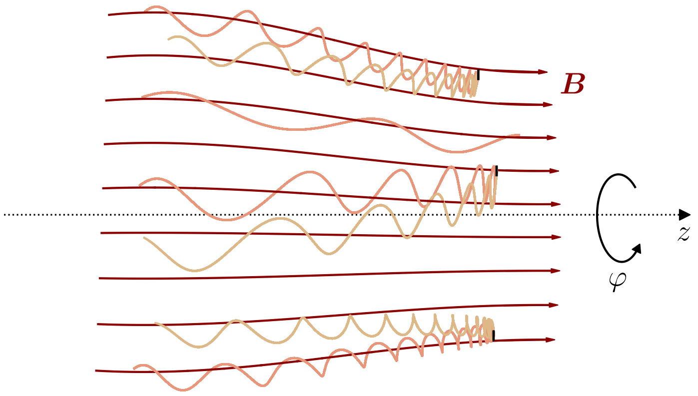
$$|\boldsymbol{b}'|^2 = 1 = \left|\begin{pmatrix}0\\0\\1\end{pmatrix} + \begin{pmatrix}\delta b_1\\\delta b_2\\\delta b_3\end{pmatrix}\right|^2 = \delta b_1^2 + \delta b_2^2 + (1+\delta b_3)^2 \sim \delta b_1^2 + \delta b_2^2 + \delta b_3^2 + 2\delta b_3 + 1. \quad (3.116)$$

$$\langle\boldsymbol{e}_\perp\times(\boldsymbol{e}_\perp\cdot\boldsymbol{\nabla})\,\boldsymbol{b}\rangle = \tfrac{1}{2}\boldsymbol{e}_3\left(\boldsymbol{e}_3\cdot\nabla\times\boldsymbol{b}\right) = \tfrac{1}{2}\boldsymbol{b}\left(\boldsymbol{b}\cdot\nabla\times\boldsymbol{b}\right) , \quad (3.117)$$

$$\left\langle\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t}\right\rangle = \overset{1}{v_\parallel}\boldsymbol{b} + \overset{2}{\boldsymbol{v}}_\mathrm{E} + \overset{3}{\frac{\epsilon}{B}\boldsymbol{b}\times D_t\left(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E}\right)} + \overset{4}{\frac{\epsilon u_\perp^2}{2B}\left(\boldsymbol{b}\cdot\boldsymbol{\nabla}\times\boldsymbol{b}\right)\boldsymbol{b}} + \overset{5}{\frac{\epsilon u_\perp^2}{2B^2}\boldsymbol{b}\times\boldsymbol{\nabla}B} + \mathcal{O}(\epsilon^2) . \quad (3.118)$$

$$\boldsymbol{v}_\mathrm{c} = \frac{mv_\parallel^2}{qB}\left[\boldsymbol{b}\times(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}\right] = \frac{mv_\parallel^2}{qB^2}\boldsymbol{B}\times\left[\boldsymbol{b}\cdot\left(\boldsymbol{e}_1\nabla b_1 + \boldsymbol{e}_2\nabla b_2 + \boldsymbol{b}\nabla b_3\right)\right] =$$
$$= \frac{mv_\parallel^2}{qB^2}\boldsymbol{B}\times\nabla b_3 = -\frac{mv_\parallel^2}{qB^2R^2}\boldsymbol{B}\times\boldsymbol{R}, \quad (3.119)$$

$$\frac{\epsilon}{B}\boldsymbol{b}\times D_t\left(v_\parallel\boldsymbol{b}+\boldsymbol{v}_\mathrm{E}\right) = \frac{\epsilon}{B}\boldsymbol{b}\times\frac{\mathrm{d}\boldsymbol{v}_\mathrm{E}}{\mathrm{d}t} + \mathcal{A} = \frac{m}{qB^2}\boldsymbol{B}\times\frac{\mathrm{d}}{\mathrm{d}t}\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} + \mathcal{A} = \frac{m}{qB^4}\boldsymbol{B}\times\left(\frac{\partial\boldsymbol{E}}{\partial t}\times\boldsymbol{B}\right) + \mathcal{B} =$$
$$= \frac{m}{qB^4}\left(\boldsymbol{B}\cdot\boldsymbol{B}\frac{\partial\boldsymbol{E}}{\partial t} - \boldsymbol{B}\cdot\frac{\partial\boldsymbol{E}}{\partial t}\boldsymbol{B}\right) + \mathcal{B} = \frac{m}{qB^2}\frac{\partial\boldsymbol{E}}{\partial t} + \mathcal{C} = \frac{\operatorname{sgn}q}{\omega_\mathrm{c}B}\frac{\partial\boldsymbol{E}}{\partial t} + \mathcal{C}, \quad (3.120)$$



$$\left\langle \frac{\mathrm{d}v_\parallel}{\mathrm{d}t} \right\rangle = \frac{E_\parallel}{\epsilon} - \frac{u_\perp^2}{2B}\nabla_\parallel B + \boldsymbol{v}_\mathrm{E} \cdot D_t \boldsymbol{b} + \mathcal{O}(\epsilon) \tag{3.121}$$

$$\left\langle \frac{\mathrm{d}u_\perp}{\mathrm{d}t} \right\rangle = \frac{v_\parallel u_\perp}{2B}\nabla_\parallel B - \frac{u_\perp}{2}(\boldsymbol{\nabla} \cdot \boldsymbol{v}_\mathrm{E} - \boldsymbol{b} \cdot \nabla_\parallel \boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon) \tag{3.122}$$

$$\left\langle \frac{\mathrm{d}\phi}{\mathrm{d}t} \right\rangle = -\frac{B}{\epsilon} - \boldsymbol{e}_2 \cdot D_t \boldsymbol{e}_1 - \frac{v_\parallel}{2}\boldsymbol{b} \cdot \boldsymbol{\nabla} \times (v_\parallel \boldsymbol{b} + \boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon) \tag{3.123}$$

3.7 Magnetic mirrors

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{r}\frac{\partial}{\partial r} r B_r + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0 \tag{3.124}$$

$$r B_r = -\int\limits_0^r r' \frac{\partial B_z}{\partial z} \mathrm{d}r'. \tag{3.125}$$

$$r B_r \sim -\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \int\limits_0^r r' \mathrm{d}r' = -\frac{1}{2} r^2 \left[\frac{\partial B_z}{\partial z}\right]_{r=0}, \tag{3.126}$$

$$B_r \sim -\frac{1}{2} r \left[\frac{\partial B_z}{\partial z}\right]_{r=0}. \tag{3.127}$$

$$F = q\boldsymbol{v} \times \boldsymbol{B} = q \begin{pmatrix} v_r \\ v_\varphi \\ v_z \end{pmatrix} \times \begin{pmatrix} B_r \\ B_\varphi \\ B_z \end{pmatrix} = q \begin{pmatrix} v_\varphi B_z - v_z B_\varphi \\ v_z B_r - v_r B_z \\ v_r B_\varphi - v_\varphi B_r \end{pmatrix} =$$
$$= q \begin{pmatrix} v_\varphi B_z^{\text{①}} \\ v_z B_r^{\text{②}} - v_r B_z^{\text{③}} \\ -v_\varphi B_r^{\text{④}} \end{pmatrix}. \tag{3.128}$$

$$F_z = -qv_\varphi B_r = \frac{1}{2}qv_\varphi r\frac{\partial B_z}{\partial z}, \tag{3.129}$$

$$F_z = -\frac{1}{2}qv_\perp \operatorname{sgn} q\frac{v_\perp m}{q\operatorname{sgn} qB}\frac{\partial B_z}{\partial z} = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z} = -\mu\frac{\partial B_z}{\partial z}, \tag{3.130}$$

$$F_\parallel = -\mu\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} = -\mu\nabla_\parallel \boldsymbol{B}, \tag{3.131}$$

$$m\frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = -\mu\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}} \tag{3.132}$$

$$m\boldsymbol{v}_\parallel \cdot \frac{\mathrm{d}\boldsymbol{v}_\parallel}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}}\cdot\boldsymbol{v}_\parallel = -\mu\frac{\partial \boldsymbol{B}}{\partial \boldsymbol{s}}\cdot\frac{\mathrm{d}\boldsymbol{s}}{\mathrm{d}t} = -\mu\frac{\mathrm{d}B}{\mathrm{d}t}. \tag{3.133}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right) = 0, \tag{3.134}$$

$$0 = -\mu\frac{\mathrm{d}B}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\mu B = -\mu\frac{\mathrm{d}B}{\mathrm{d}t} + \mu\frac{\mathrm{d}B}{\mathrm{d}t} + B\frac{\mathrm{d}\mu}{\mathrm{d}t} \tag{3.135}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0. \tag{3.136}$$

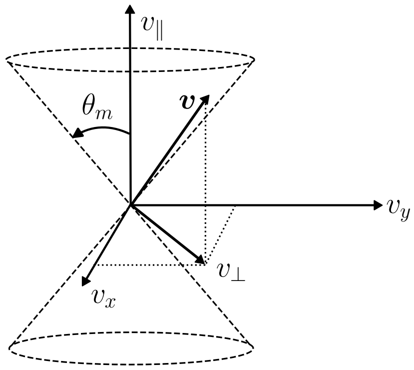
$$\frac{\mathrm{d}u_\perp}{\mathrm{d}t} = \frac{v_\parallel u_\perp}{2B}\boldsymbol{b}\cdot\nabla_\parallel B - \frac{u_\perp}{2}(\boldsymbol{\nabla}\overset{\text{A}}{\cdot}\boldsymbol{v}_\mathrm{E} - \boldsymbol{b}\cdot\overset{\text{B}}{\nabla_\parallel}\boldsymbol{v}_\mathrm{E}) + \mathcal{O}(\epsilon), \tag{3.137}$$

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\boldsymbol{v}_\mathrm{E} &= \boldsymbol{\nabla}\cdot\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} = \frac{1}{B^2}\left[\boldsymbol{B}\cdot(\boldsymbol{\nabla}\times\boldsymbol{E}) - \boldsymbol{E}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B})\right] + (\boldsymbol{E}\times\boldsymbol{B})\cdot\nabla\frac{1}{B^2} = \\
&= -\frac{1}{B^2}\boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t} - \frac{\boldsymbol{E}_\parallel + \boldsymbol{E}_\perp}{B^2}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - \frac{2(\boldsymbol{E}\times\boldsymbol{B})}{B^2}\cdot\frac{\nabla B}{B} = \\
&= -\frac{B}{B^2}\boldsymbol{b}\cdot\left[B\frac{\partial\boldsymbol{b}}{\partial t} + \boldsymbol{b}\frac{\partial B}{\partial t}\right] - \frac{\boldsymbol{E}_\parallel}{B^2}\cdot\left[(\nabla B)\times\boldsymbol{b} + B(\boldsymbol{\nabla}\times\boldsymbol{b})\right] - \\
&- \frac{\boldsymbol{E}_\perp}{B^2}\cdot(\boldsymbol{\nabla}\times\boldsymbol{B}) - 2\boldsymbol{v}_\mathrm{E}\cdot\frac{\nabla B}{B} = -\boldsymbol{b}\cdot\frac{\partial\boldsymbol{b}}{\partial t} - \frac{1}{B}\frac{\partial B}{\partial t} - \\
&- \frac{E_\parallel}{B^2}\boldsymbol{b}\cdot\left[(\nabla B)\times\boldsymbol{b}\right] - \frac{E_\parallel}{B}\boldsymbol{b}\cdot(\boldsymbol{\nabla}\times\boldsymbol{b}),
\end{aligned} \tag{3.138}$$

$$\begin{aligned}
\boldsymbol{b}\cdot\nabla_\parallel\boldsymbol{v}_\mathrm{E} &= \nabla_\parallel(\boldsymbol{b}\cdot\boldsymbol{v}_\mathrm{E}) - \boldsymbol{v}_\mathrm{E}\cdot(\nabla_\parallel\boldsymbol{b}) = -\boldsymbol{v}_\mathrm{E}\cdot\left[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{b}\right] = \\
&- \frac{\boldsymbol{v}_\mathrm{E}}{B}\cdot\left[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{B}\right] - (\boldsymbol{v}_\mathrm{E}\cdot\boldsymbol{B})\,(\boldsymbol{b}\cdot\boldsymbol{\nabla})\frac{1}{B} = -\frac{\boldsymbol{v}_\mathrm{E}}{B}\cdot\left[(\boldsymbol{b}\cdot\boldsymbol{\nabla})\boldsymbol{B}\right],
\end{aligned} \tag{3.139}$$

$$\nabla B^2 = \nabla(\boldsymbol{B}\cdot\boldsymbol{B}) = 2\boldsymbol{B}\cdot\nabla\boldsymbol{B} = 2B\boldsymbol{b}\cdot\nabla\boldsymbol{B} \quad \text{and} \quad \nabla B^2 = 2B\nabla B. \tag{3.140}$$

$$\begin{aligned}
\frac{\mathrm{d}u_\perp}{\mathrm{d}t} &= \frac{v_\parallel u_\perp}{2B}\boldsymbol{b}\cdot\nabla_\parallel B + \frac{u_\perp}{2}\frac{1}{B}\frac{\partial B}{\partial t} + \frac{u_\perp}{2}\boldsymbol{v}_\mathrm{E}\cdot\frac{\nabla B}{B} + \mathcal{O}(\epsilon) = \\
&= \frac{u_\perp}{2}\left[\frac{\partial}{\partial t} + (v_\parallel\boldsymbol{b} + \boldsymbol{v}_\mathrm{E})\cdot\boldsymbol{\nabla}\right]B + \mathcal{O}(\epsilon) = \frac{u_\perp}{2B}D_t B + \mathcal{O}(\epsilon),
\end{aligned} \tag{3.141}$$



$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = \frac{u_\perp \frac{\mathrm{d}}{\mathrm{d}t} u_\perp}{B} + \frac{u_\perp^2}{2B} \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{B} = \frac{u_\perp^2}{2B^2} D_t B + \mathcal{O}(\epsilon) - \frac{u_\perp^2}{2B^2} \frac{\mathrm{d}B}{\mathrm{d}t} = \mathcal{O}(\epsilon). \tag{3.142}$$

$$\mu = \mathrm{const} = \frac{1}{2} m v_{\perp,0}^2 / B_0 = \frac{1}{2} m v_\perp' / B' \quad \text{and} \quad v_\perp'^2 = v_{\perp,0}^2 + v_{\parallel,0}^2 \equiv v_0^2, \tag{3.143}$$

$$\frac{B_0}{B'} = \frac{v_{\perp,0}^2}{v_\perp'^2} = \frac{v_{\perp,0}^2}{v_0^2} = \sin^2 \vartheta, \tag{3.144}$$

$$\sin^2 \vartheta_{\max} = \frac{B_0}{B_{\max}}. \tag{3.145}$$

3.8 Adiabatic invariants

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}} \quad \text{and} \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{q}}. \tag{3.146}$$

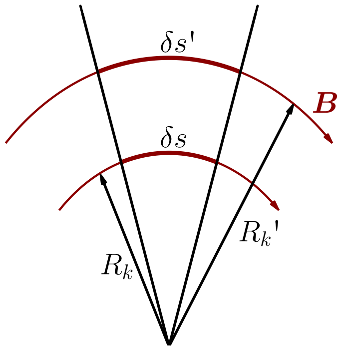
$$q_i(t) = q_i(t+T), \tag{3.147}$$

$$\oint p_i \mathrm{d}q_i = \mathrm{const} \tag{3.148}$$

$$\int_0^T p \mathrm{d}q = \int_0^T (-ml^2 \varphi_0 \omega \sin \omega t)(-\varphi_0 \omega \sin \omega t) \mathrm{d}(\omega t) =$$
$$= ml^2 \varphi_0^2 \omega \int_0^T \sin^2 \omega t \mathrm{d}(\omega t) = \pi m l^2 \varphi_0^2 \omega = \mathrm{const}. \tag{3.149}$$

$$J_1 = \oint p_i \mathrm{d}q_i = \int_0^{2\pi} m v_\perp r_\mathrm{L} \, \mathrm{d}\phi = 2\pi v_\perp \frac{m^2 v_\perp}{q \operatorname{sgn} q B} = \frac{4\pi m}{q \operatorname{sgn} q} \mu = \mathrm{const}. \tag{3.150}$$

$$J_2 = \int_a^b v_\parallel \mathrm{d}s \tag{3.151}$$



$$\frac{\delta s}{R_k} = \frac{\delta s'}{R'_k}. \tag{3.152}$$

$$\frac{\delta s' - \delta s}{\Delta t \delta s} = \frac{R'_k - R_k}{\Delta t R_k} = \boldsymbol{v}_{\mathrm{gc}} \cdot \frac{\boldsymbol{R}_k}{R_k^2} = \frac{1}{\delta s}\frac{\Delta \delta s}{\Delta t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t}, \tag{3.153}$$

$$\left| \frac{R'_k - R_k}{\Delta t} \right| = \boldsymbol{v}_{\mathrm{gc}} \cdot \frac{\boldsymbol{R}_k}{R_k}. \tag{3.154}$$

$$\boldsymbol{v}_{\mathrm{gc}} = \frac{1}{2} \operatorname{sgn} q v_\perp r_{\mathrm{L}} \frac{\boldsymbol{B} \times \nabla B}{B^2} + \frac{m v_\parallel^2}{q} \frac{\boldsymbol{R_k} \times \boldsymbol{B}}{R_k^2 B^2}. \tag{3.155}$$

$$\frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} = \frac{1}{2}\frac{m}{q}\frac{v_\perp^2}{B^3}(\boldsymbol{B} \times \nabla B) \cdot \frac{\boldsymbol{R}_k}{R_k^2}, \tag{3.156}$$

$$W = \frac{1}{2} m v_\parallel^2 + \frac{1}{2} m v_\perp^2 = \frac{1}{2} m v_\parallel^2 + \mu B = W_\parallel + W_\perp. \tag{3.157}$$

$$\frac{\mathrm{d}v_\parallel}{\mathrm{d}t} = \frac{1}{2}\frac{1}{\sqrt{\frac{2}{m}(W - \mu B)}}\left(-\mu\frac{\mathrm{d}B}{\mathrm{d}t}\right)\frac{2}{m} \tag{3.158}$$

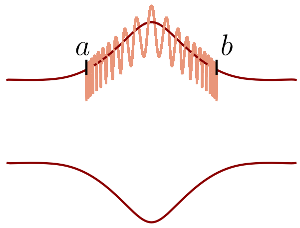
$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{1}{2}\frac{\mu\frac{\mathrm{d}B}{\mathrm{d}t}}{W - \mu B} = -\frac{\mu\frac{\mathrm{d}B}{\mathrm{d}t}}{m v_\parallel^2}, \tag{3.159}$$

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\mathrm{d}B}{\mathrm{d}R}\frac{\mathrm{d}R}{\mathrm{d}t} = \boldsymbol{v}_{\mathrm{gc}} \cdot \nabla B = \frac{m v_\parallel^2}{q}\frac{\boldsymbol{R}_k \times \boldsymbol{B}}{R_k^2 B^2} \cdot \nabla B. \tag{3.160}$$

$$\frac{\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}}{v_\parallel} = -\frac{\mu}{q}\frac{(\boldsymbol{R}_k \times \boldsymbol{B}) \cdot \nabla B}{R_k^2 B^2} = -\frac{1}{2}\frac{m v_\perp^2}{Bq}\frac{(\boldsymbol{B} \times \nabla B) \cdot \boldsymbol{R}_k}{R_k^2 B^2}. \tag{3.161}$$

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = \frac{1}{\delta s}\frac{\mathrm{d}\delta s}{\mathrm{d}t} + \frac{1}{v_\parallel}\frac{\mathrm{d}v_\parallel}{\mathrm{d}t}. \tag{3.162}$$

$$\frac{1}{v_\parallel \delta s}\frac{\mathrm{d}(v_\parallel \delta s)}{\mathrm{d}t} = 0 \tag{3.163}$$



$$J_2 = \int_a^{a'} v_\parallel \delta s + \int_{a'}^{b'} v_\parallel \delta s + \int_{b'}^{b} v_\parallel \delta s, \tag{3.164}$$

$$J_2 = \langle v_\parallel \rangle L, \tag{3.165}$$

$$J_3 = \Phi = \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \tag{3.166}$$

3.9 van Allen radiation belts

$$\boldsymbol{B} = \left(B_R(R,z), 0, B_z(R,z)\right), \tag{3.167}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 = \frac{1}{R}\frac{\partial}{\partial R} R B_R + \frac{\partial B_z}{\partial z}. \tag{3.168}$$

$$B_z = \frac{1}{R}\frac{\partial F}{\partial R} \quad \text{and} \quad B_R = -\frac{1}{R}\frac{\partial F}{\partial z}. \tag{3.169}$$

$$F = \mathcal{M}\frac{R^2}{(R^2+z^2)^{3/2}}, \tag{3.170}$$

$$\left(m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\varphi = q\left(v_z B_R - v_R B_z\right) = -\frac{q}{R}\left(v_z \frac{\partial F}{\partial z} + v_R \frac{\partial F}{\partial R}\right) = -\frac{q}{R}\frac{\mathrm{d}F}{\mathrm{d}t}, \tag{3.171}$$

$$\boldsymbol{e}_\varphi \propto \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.172}$$

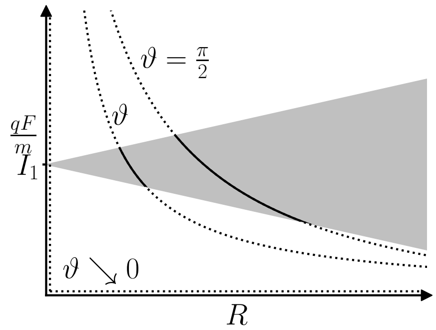
$$|z||r| \sin\vartheta = R, \tag{3.173}$$

$$R\boldsymbol{e}_\varphi = \boldsymbol{e}_z \times \boldsymbol{r}. \tag{3.174}$$

$$R v_\varphi = R\boldsymbol{e}_\varphi \cdot \boldsymbol{v} = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \boldsymbol{v}. \tag{3.175}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} R v_\varphi = (\boldsymbol{e}_z \times \boldsymbol{r}) \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} + (\boldsymbol{e}_z \times \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}) \cdot \boldsymbol{v}. \tag{3.176}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} R v_\varphi = R\boldsymbol{e}_\varphi \cdot \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = R\left(\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}\right)_\varphi. \tag{3.177}$$



$$\frac{\mathrm{d}}{\mathrm{d}t}\left(Rv_\varphi + \frac{qF}{m}\right) = R\frac{\mathrm{d}v_\varphi}{\mathrm{d}t} + \frac{q}{m}\frac{\mathrm{d}F}{\mathrm{d}t} = 0, \tag{3.178}$$

$$Rv_\varphi + \frac{qF}{m} = I_1 \tag{3.179}$$

$$v_R^2 + v_\varphi^2 + v_z^2 = v^2 = I_2^2 \tag{3.180}$$

$$R|v_\varphi| \leq |v|R \quad \rightarrow \quad \left|I_1 - \frac{qF}{m}\right| \leq RI_2. \tag{3.181}$$

Chapter 4

Plasma as fluid

4.1 Double-fluid model in physics of plasmas

$$\frac{\partial n_\alpha}{\partial t} + \boldsymbol{\nabla} \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0 \tag{4.1}$$

$$m_\alpha n_\alpha \left[\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + (\boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla}) \boldsymbol{u}_\alpha \right] = q_\alpha n_\alpha (\boldsymbol{E} + \boldsymbol{u}_\alpha \times \boldsymbol{B}) - \nabla p_\alpha \tag{4.2}$$

$$p_\alpha = C_\alpha (m_\alpha n_\alpha)^{\gamma_\alpha}, \tag{4.3}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = n_\mathrm{i} q_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \tag{4.4}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{4.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{4.6}$$

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \boldsymbol{B} = n_\mathrm{i} q_\mathrm{i} \boldsymbol{u}_\mathrm{i} + n_\mathrm{e} q_\mathrm{e} \boldsymbol{u}_\mathrm{e} + \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{4.7}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial}{\partial t} \boldsymbol{\nabla} \cdot \boldsymbol{B}. \tag{4.8}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \mathrm{const}, \tag{4.9}$$

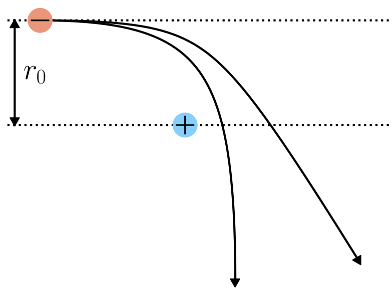
$$p = C\rho^\gamma \quad \text{or} \quad \frac{\nabla p}{p} = \gamma \frac{\nabla \rho}{\rho}. \tag{4.10}$$

4.2 Highly ionised plasmas

$$m_\mathrm{e} n_\mathrm{e} \frac{\mathrm{d}\boldsymbol{u}_\mathrm{e}}{\mathrm{d}t} = -e n_\mathrm{e} (\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - \nabla p_\mathrm{e} + \boldsymbol{P}_\mathrm{ei}. \tag{4.11}$$

$$\boldsymbol{P}_\mathrm{ei} = -\boldsymbol{P}_\mathrm{ie}. \tag{4.12}$$

$$\boldsymbol{P}_\mathrm{ei} = \underbrace{\eta}_{5} \underbrace{e^2}_{1} \underbrace{n_\mathrm{e}}_{2} \underbrace{n_\mathrm{i} \underbrace{(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e})}_{3}}_{4} \sim \eta e^2 n^2 (\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}), \tag{4.13}$$



$$\boldsymbol{P}_{\mathrm{ei}} = m_{\mathrm{e}} n (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) \nu_{\mathrm{ei}}, \tag{4.14}$$

$$\nu_{\mathrm{ei}} = \frac{ne^2}{m_{\mathrm{e}}} \eta = \omega_{\mathrm{p}}^2 \varepsilon_0 \eta, \tag{4.15}$$

$$F = -\frac{e^2}{4\pi\varepsilon_0 r^2}, \tag{4.16}$$

$$\Delta t = \frac{r_0}{v}, \tag{4.17}$$

$$\Delta(m_{\mathrm{e}} v) \sim |F\, \Delta t| = \frac{e^2}{4\pi\varepsilon_0 r_0 v}. \tag{4.18}$$

$$\Delta(m_{\mathrm{e}} v) = m_{\mathrm{e}} v = \frac{e^2}{4\pi\varepsilon_0 r_0 v}, \tag{4.19}$$

$$r_{90} = \frac{e^2}{4\pi\varepsilon_0 m_{\mathrm{e}} v^2}. \tag{4.20}$$

$$\sigma = \pi r_{90}^2 = \frac{e^4}{16\pi\varepsilon_0^2 m_{\mathrm{e}}^2 v^4}. \tag{4.21}$$

$$\nu_{\mathrm{ei}} = \sigma n v = \frac{ne^4}{16\pi\varepsilon_0^2 m_{\mathrm{e}}^2 v^3}. \tag{4.22}$$

$$\eta = \frac{\nu_{\mathrm{ei}} m_{\mathrm{e}}}{ne^2} = \frac{e^2}{16\pi\varepsilon_0^2 m_{\mathrm{e}} v^3} = \frac{e^2 m_{\mathrm{e}}^{1/2}}{16\pi\varepsilon_0^2 (K_{\mathrm{B}} T_{\mathrm{e}})^{3/2}}. \tag{4.23}$$

$$\int_{r_{90}}^{\lambda_{\mathrm{D}}} \frac{1}{r} \mathrm{d}r = [\ln r]_{r_{90}}^{\lambda_{\mathrm{D}}} = \ln \frac{\lambda_{\mathrm{D}}}{r_{90}} = \ln \Lambda, \tag{4.24}$$

$$\eta = \frac{e^2 m_{\mathrm{e}}^{1/2}}{16\pi\varepsilon_0^2 (K_{\mathrm{B}} T_{\mathrm{e}})^{3/2}} \ln \Lambda. \tag{4.25}$$

$$\boldsymbol{P}_{\mathrm{ei}} = \eta e^2 n^2 (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) = e n_{\mathrm{e}} \boldsymbol{E}. \tag{4.26}$$

$$\boldsymbol{E} = \eta e n (\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) = \eta \boldsymbol{j}, \tag{4.27}$$

4.3 Single-fluid model

$$\rho \equiv n_\mathrm{i}M + n_\mathrm{e}m \sim n(M+m) \sim nM, \tag{4.28}$$

$$\boldsymbol{u} \equiv \frac{1}{\rho}(n_\mathrm{i}M\boldsymbol{u}_\mathrm{i} + n_\mathrm{e}m\boldsymbol{u}_\mathrm{e}) \sim \frac{n(M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e})}{n(M+m)} = \frac{M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e}}{M+m}, \tag{4.29}$$

$$\boldsymbol{j} \equiv e(n_\mathrm{i}\boldsymbol{u}_\mathrm{i} - n_\mathrm{e}\boldsymbol{u}_\mathrm{e}) \sim ne(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}), \tag{4.30}$$

$$p = p_\mathrm{i} + p_\mathrm{e}. \tag{4.31}$$

$$Mn\frac{\partial \boldsymbol{u}_\mathrm{i}}{\partial t} = en(\boldsymbol{E} + \boldsymbol{u}_\mathrm{i} \times \boldsymbol{B}) - \nabla p_\mathrm{i} + Mn\boldsymbol{g} + \boldsymbol{P}_\mathrm{ie}, \tag{4.32}$$

$$mn\frac{\partial \boldsymbol{u}_\mathrm{e}}{\partial t} = -en(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - \nabla p_\mathrm{e} + mn\boldsymbol{g} + \boldsymbol{P}_\mathrm{ei} \tag{4.33}$$

$$n(M+m)\frac{\partial}{\partial t}\left(\frac{M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e}}{M+m}\right) = n\frac{\partial}{\partial t}(M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e}) = en(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) \times \boldsymbol{B} - \nabla p + n(M+m)\boldsymbol{g}. \tag{4.34}$$

$$\rho\frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho\boldsymbol{g}, \tag{4.35}$$

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{i}}{\partial t} = emn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{i} \times \boldsymbol{B}) - m\nabla p_\mathrm{i} + mMn\boldsymbol{g} + m\boldsymbol{P}_\mathrm{ie}, \tag{4.36}$$

$$mMn\frac{\partial \boldsymbol{u}_\mathrm{e}}{\partial t} = -eMn(\boldsymbol{E} + \boldsymbol{u}_\mathrm{e} \times \boldsymbol{B}) - M\nabla p_\mathrm{e} + mMn\boldsymbol{g} + M\boldsymbol{P}_\mathrm{ei}. \tag{4.37}$$

$$\begin{aligned} nmM\frac{\partial}{\partial t}(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = {} & en(M+m)\boldsymbol{E} + en(m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e}) \times \boldsymbol{B} - \\ & - m\nabla p_\mathrm{i} + M\nabla p_\mathrm{e} - (M+m)\boldsymbol{P}_\mathrm{ei}. \end{aligned} \tag{4.38}$$

$$m\boldsymbol{u}_\mathrm{i} + M\boldsymbol{u}_\mathrm{e} = M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e} + M(\boldsymbol{u}_\mathrm{e} - \boldsymbol{u}_\mathrm{i}) + m(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \frac{\rho}{n}\boldsymbol{u} - (M-m)\frac{\boldsymbol{j}}{ne}, \tag{4.39}$$

$$P_\mathrm{ei} = \eta e^2 n^2(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}) = \eta en\boldsymbol{j}. \tag{4.40}$$

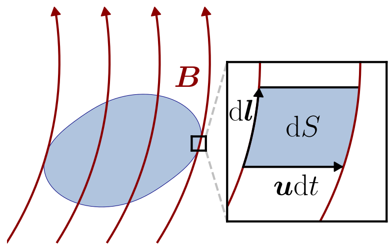
$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{e\rho}\left[\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} + (M-m)\boldsymbol{j} \times \boldsymbol{B} + m\nabla p_\mathrm{i} - M\nabla p_\mathrm{e}\right]. \tag{4.41}$$

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\frac{m}{eB}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} = MnB\omega_\mathrm{c}^{-1}\frac{\partial}{\partial t}\frac{\boldsymbol{j}}{n} \ll (M-m)\boldsymbol{j} \times \boldsymbol{B}, \tag{4.42}$$

$$\frac{1}{\omega_\mathrm{c}}\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\boldsymbol{j}}{n}\right) \ll \frac{\boldsymbol{j}}{n} \tag{4.43}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} - \eta\boldsymbol{j} = \frac{1}{en}\left(\boldsymbol{j} \times \boldsymbol{B} - \nabla p_\mathrm{e}\right), \tag{4.44}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho\boldsymbol{u} = 0 \tag{4.45}$$



$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.46}$$

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p + \rho \boldsymbol{g}, \tag{4.47}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j}, \tag{4.48}$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{u} = 0, \tag{4.49}$$

$$\frac{\partial \rho_{\mathrm{e}}}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0. \tag{4.50}$$

4.4 Approximations to plasma fluid

$$f_{\mathrm{Lorentz}} = \rho_e \boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.51}$$

$$\boldsymbol{j} \times \boldsymbol{B} = 0. \tag{4.52}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu \boldsymbol{j}. \tag{4.53}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = 0, \tag{4.54}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \alpha \boldsymbol{B}. \tag{4.55}$$

$$\boldsymbol{E'} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \boldsymbol{0}. \tag{4.56}$$

$$\boldsymbol{E'} = \eta \boldsymbol{j}, \tag{4.57}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \nabla \boldsymbol{\nabla} \cdot \boldsymbol{B} - \triangle \boldsymbol{B} = \boldsymbol{\nabla} \times \left(\mu_0 \boldsymbol{j} + \mu_0 \varepsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \right). \tag{4.58}$$

$$\sigma \mu_0 \boldsymbol{\nabla} \times (\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = -\varepsilon_0 \mu_0 \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}}{\partial t} - \triangle \boldsymbol{B}. \tag{4.59}$$

$$-\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = 0 \quad \to \quad \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t}. \tag{4.60}$$

$$\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = \int \boldsymbol{B} \cdot \frac{\mathrm{d}\boldsymbol{S}}{\mathrm{d}t} + \int \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{4.61}$$

$$\oint_l \boldsymbol{B}\cdot(\boldsymbol{u}\times \mathrm{d}\boldsymbol{l}) + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S} = -\oint_l (\boldsymbol{u}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{l} + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S}. \tag{4.62}$$

$$-\int \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B})\cdot \mathrm{d}\boldsymbol{S} + \int \frac{\partial \boldsymbol{B}}{\partial t}\cdot \mathrm{d}\boldsymbol{S} = -\int \left[\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) - \frac{\partial \boldsymbol{B}}{\partial t}\right]\cdot \mathrm{d}\boldsymbol{S} = \boldsymbol{0}, \tag{4.63}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E} = \left(\frac{1}{R}\frac{\partial E_z}{\partial\varphi} - \frac{\partial E_R}{\partial z}, \frac{\partial E_R}{\partial z} - \frac{\partial E_z}{\partial R}, \frac{1}{R}\frac{\partial}{\partial R}RE_\varphi - \frac{1}{R}\frac{\partial E_R}{\partial \varphi}\right) = \boldsymbol{0}. \tag{4.64}$$

$$E_\varphi = -(\boldsymbol{u}\times\boldsymbol{B})_\varphi = -u_R B_z + u_z B_R = 0, \tag{4.65}$$

$$\frac{u_R}{u_z} = \frac{B_R}{B_z}. \tag{4.66}$$

$$\boldsymbol{u}_\mathrm{p} = \xi(R,z)\boldsymbol{B}_\mathrm{p}, \tag{4.67}$$

$$\Psi(R,z) = RA_\varphi, \tag{4.68}$$

$$B_R = -\frac{1}{R}\frac{\partial\Psi}{\partial z} \quad \text{and} \quad B_z = \frac{1}{R}\frac{\partial\Psi}{\partial R}. \tag{4.69}$$

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}_\mathrm{p}) + \frac{\partial\rho v_\varphi}{\partial\varphi} = \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}_\mathrm{p}) = 0, \tag{4.70}$$

$$\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}_\mathrm{p}) = \boldsymbol{\nabla}\cdot(\rho\xi\boldsymbol{B}_\mathrm{p}) = \nabla\rho\xi\cdot\boldsymbol{B}_\mathrm{p} + \rho\xi\boldsymbol{\nabla}\cdot\boldsymbol{B}_\mathrm{p} = 0. \tag{4.71}$$

$$\boldsymbol{\nabla}\cdot(\rho\xi\boldsymbol{B}_\mathrm{p}) = \nabla_R(\rho\xi)B_R + \nabla_z(\rho\xi)B_z = -\frac{1}{R}\nabla_R(\rho\xi)\nabla_z\Psi + \frac{1}{R}\nabla_z(\rho\xi)\nabla_R\Psi = 0, \tag{4.72}$$

$$\nabla(\rho\xi)\times\nabla\Psi = \boldsymbol{0}. \tag{4.73}$$

$$4\pi\rho\xi = F_1(\Psi). \tag{4.74}$$

$$\begin{aligned}\boldsymbol{u}\times\boldsymbol{B} &= u_\varphi\boldsymbol{e}_\varphi\times\boldsymbol{B}_\mathrm{p} + \boldsymbol{u}_\mathrm{p}\times B_\varphi\boldsymbol{e}_\varphi = u_\varphi\boldsymbol{e}_\varphi\times\boldsymbol{B}_\mathrm{p} + \xi\boldsymbol{B}_\mathrm{p}\times B_\varphi\boldsymbol{e}_\varphi \\ &= (u_\varphi - \xi B_\varphi)(B_z - B_R)\boldsymbol{e}_z = (u_\varphi - \xi B_\varphi)\frac{1}{R}\left(\frac{\partial\Psi}{\partial R} - \frac{\partial\Psi}{\partial z}\right)\boldsymbol{e}_z \\ &= \frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi. \end{aligned} \tag{4.75}$$

$$\boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}) = \frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{0}, \tag{4.76}$$

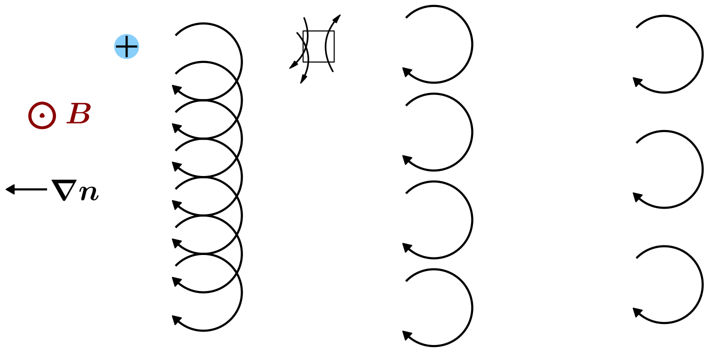
$$\nabla\times\left(\frac{u_\varphi - \xi B_\varphi}{R}\nabla\Psi\right) = \nabla\left(\frac{u_\varphi - \xi B_\varphi}{R}\right)\times\nabla\Psi = \boldsymbol{0}, \tag{4.77}$$

$$\frac{u_\varphi - \xi B_\varphi}{R} = F_2(\Psi). \tag{4.78}$$

$$B_\varphi\boldsymbol{e}_\varphi\cdot\nabla(RB_\varphi - F_1 R v_\varphi) = 0, \tag{4.79}$$

$$RB_\varphi - F_1(\Psi)Rv_\varphi = F_3(\Psi). \tag{4.80}$$

$$\frac{1}{2}u^2 + \int_{\Psi=\mathrm{const}}\frac{\mathrm{d}P}{\rho} + \Phi - Ru_\varphi F_3(\Psi) = F_4(\Psi). \tag{4.81}$$



4.5 Drifts in the plasma fluid

$$mn\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.82}$$

$$0 = qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - \nabla p \tag{4.83}$$

$$\begin{aligned} 0 &= qn[\boldsymbol{E}\times\boldsymbol{B} + (\boldsymbol{u}\times\boldsymbol{B})\times\boldsymbol{B}] - \nabla p\times\boldsymbol{B} = \\ &= qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}B^2 + \boldsymbol{B}\boldsymbol{u}\cdot\boldsymbol{B}) - \nabla p\times\boldsymbol{B}. \end{aligned} \tag{4.84}$$

$$qn(\boldsymbol{E}\times\boldsymbol{B} - \boldsymbol{u}_\perp B^2) - \nabla p\times\boldsymbol{B} = 0 \tag{4.85}$$

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} - \frac{\nabla p\times\boldsymbol{B}}{qnB^2} = \boldsymbol{v}_\mathrm{E} + \boldsymbol{v}_\mathrm{D}. \tag{4.86}$$

$$mn\left(\frac{\partial u_z}{\partial t} + (\boldsymbol{u}\cdot\nabla)u_z\right) = qnE_z - \frac{\partial p}{\partial z} \tag{4.87}$$

$$\frac{\nabla p}{p} = \gamma\frac{\nabla n}{n}, \tag{4.88}$$

$$mn\frac{\partial u_z}{\partial t} = qnE_z - K_\mathrm{B}T\frac{\partial n}{\partial z}. \tag{4.89}$$

$$e\frac{\partial\phi}{\partial z} = \frac{K_\mathrm{B}T_\mathrm{e}}{n}\frac{\partial n}{\partial z} \tag{4.90}$$

$$e\phi = K_\mathrm{B}T_\mathrm{e}\ln n + \mathcal{C} \tag{4.91}$$

$$n = n_0\exp\left[\frac{e\phi}{K_\mathrm{B}T_\mathrm{e}}\right], \tag{4.92}$$

Chapter 5

Waves in plasmas

5.1 Linear waves

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}, \tag{5.1}$$

$$\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{u}_1. \tag{5.2}$$

5.2 Example: Acoustic waves in fluids

$$\rho \left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \right] = -\nabla p = -\frac{\gamma p}{\rho} \nabla \rho, \tag{5.3}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{5.4}$$

$$(\rho_0 + \rho_1) \left\{ \frac{\partial \boldsymbol{u}_0}{\partial t} + \frac{\partial \boldsymbol{u}_1}{\partial t} + \left[(\boldsymbol{u}_0 + \boldsymbol{u}_1) \cdot \boldsymbol{\nabla} \right] (\boldsymbol{u}_0 + \boldsymbol{u}_1) \right\} = -\frac{\gamma (p_0 + p_1)}{(\rho_0 + \rho_1)} \nabla (\rho_0 + \rho_1) \tag{5.5}$$

$$\frac{\partial (\rho_0 + \rho_1)}{\partial t} + \nabla \cdot \left[(\rho_0 + \rho_1)(\boldsymbol{u}_0 + \boldsymbol{u}_1) \right] = 0. \tag{5.6}$$

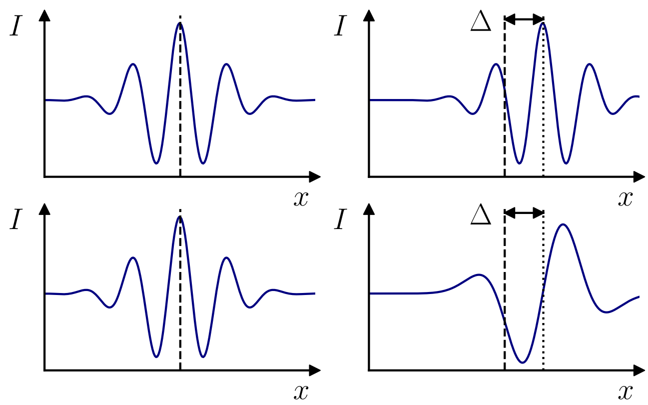
$$\overset{\textcircled{1}}{\frac{\partial \rho_0}{\partial t}} + \overset{\textcircled{2}}{\frac{\partial \rho_1}{\partial t}} + \nabla \cdot \overset{\textcircled{3}}{(\rho_0 \boldsymbol{u}_0)} + \rho_0 \overset{\textcircled{4}}{\nabla} \cdot \boldsymbol{u}_1 + \boldsymbol{u}_1 \overset{\textcircled{5}}{\cdot} \nabla \rho_0 + \nabla \cdot \overset{\textcircled{6}}{(\rho_1 \boldsymbol{u}_0)} + \nabla \cdot \overset{\textcircled{7}}{(\rho_1 \boldsymbol{u}_1)} = 0. \tag{5.7}$$

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_1 = 0 \tag{5.8}$$

$$\frac{p_0 + p_1}{\rho_0 + \rho_1} = \frac{\frac{p_0 + p_1}{\rho_0}}{\frac{\rho_0 + \rho_1}{\rho_0}} = \frac{\frac{p_0}{\rho_0} + \frac{p_1}{\rho_0}}{1 + \frac{\rho_1}{\rho_0}} = \frac{p_0}{\rho_0},$$

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} = -\gamma \frac{p_0}{\rho_0} \nabla \rho_1. \tag{5.9}$$

$$A_1 = \tilde{A}_1 \exp \left[\mathrm{i} (\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) \right], \tag{5.10}$$



$$\frac{\partial \bullet}{\partial t} \to -\mathrm{i}\omega \bullet \tag{5.11}$$

$$\nabla \bullet \to \mathrm{i}\boldsymbol{k} \bullet \tag{5.12}$$

$$\triangle \bullet \to -k^2 \bullet \tag{5.13}$$

$$-\mathrm{i}\omega\rho_1 + \mathrm{i}\rho_0 \boldsymbol{k} \cdot \boldsymbol{u}_1 = 0, \tag{5.14}$$

$$-\mathrm{i}\omega\rho_0 \boldsymbol{u}_1 + \mathrm{i}\gamma \frac{p_0}{\rho_0} \boldsymbol{k} \rho_1 = 0. \tag{5.15}$$

$$\frac{\omega}{k} = \sqrt{\frac{\gamma p_0}{\rho_0}} \equiv c_{\mathrm{s}}, \tag{5.16}$$

$$-\mathrm{i}\omega\rho_1 + \mathrm{i}\rho_1 \boldsymbol{k} \cdot \boldsymbol{u}_0 + \mathrm{i}\rho_0 \boldsymbol{k} \cdot \boldsymbol{u}_1 = 0, \tag{5.17}$$

$$-\mathrm{i}\rho_1(\omega - \boldsymbol{k} \cdot \boldsymbol{u}_0) + \mathrm{i}\rho_0 \boldsymbol{k} \cdot \boldsymbol{u}_1 = 0. \tag{5.18}$$

$$\rho_0 \frac{\partial \boldsymbol{u}_1}{\partial t} + \rho_0(\boldsymbol{u}_0 \cdot \boldsymbol{\nabla} \boldsymbol{u}_1) = -\gamma \frac{p_0}{\rho_0} \boldsymbol{\nabla} \rho_1, \tag{5.19}$$

$$-\mathrm{i}\rho_0(\omega - \boldsymbol{k} \cdot \boldsymbol{u}_0)\boldsymbol{u}_1 + \mathrm{i}k\gamma \frac{p_0}{\rho_0} \rho_1 = 0, \tag{5.20}$$

$$\omega - \boldsymbol{k} \cdot \boldsymbol{u}_0 = k c_{\mathrm{s}}, \tag{5.21}$$

5.3 Types of plasma waves

5.4 Electrostatic waves without a background magnetic field

$$mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0 \boldsymbol{E}_1, \tag{5.22}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + \boldsymbol{\nabla} \cdot [n_0 \boldsymbol{u}_{\mathrm{e}1}] = 0, \tag{5.23}$$

$$\boldsymbol{E}_1 = -\nabla \phi_1, \tag{5.24}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = -en_{\mathrm{e}1}. \tag{5.25}$$

$$\varepsilon_0 \triangle \phi_1 = n_{\mathrm{e}1} e, \tag{5.26}$$

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = en_0 \triangle \phi_1, \tag{5.27}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = 0. \tag{5.28}$$

$$mn_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = \frac{e^2 n_0}{\varepsilon_0} n_{\mathrm{e}1}, \tag{5.29}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = -\frac{1}{n_0} \frac{\partial n_{\mathrm{e}1}}{\partial t}. \tag{5.30}$$

$$-m \frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0} n_{\mathrm{e}1}. \tag{5.31}$$

$$\omega^2 n_{\mathrm{e}1} = \frac{e^2 n_0}{\varepsilon_0 m} n_{\mathrm{e}1}. \tag{5.32}$$

$$\omega_{\mathrm{p}}^2 = \frac{n_0 e^2}{\varepsilon_0 m}. \tag{5.33}$$

$$\nabla p_{\mathrm{e}} = \nabla(3K_{\mathrm{B}} T_{\mathrm{e}} n_{\mathrm{e}}) = 3K_{\mathrm{B}} T_{\mathrm{e}} \nabla(n_0 + n_{\mathrm{e}1}), \tag{5.34}$$

$$mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -en_0 \boldsymbol{E}_1 - 3K_{\mathrm{B}} T_{\mathrm{e}} \nabla n_{\mathrm{e}1}, \tag{5.35}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.36}$$

$$\boldsymbol{E}_1 = -\nabla \phi_1, \tag{5.37}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = -en_{\mathrm{e}1}. \tag{5.38}$$

$$-m \frac{\partial^2 n_{\mathrm{e}1}}{\partial t^2} = \frac{e^2 n_0}{\varepsilon_0} n_{\mathrm{e}1} - 3K_{\mathrm{B}} T_{\mathrm{e}} \triangle n_{\mathrm{e}1}. \tag{5.39}$$

$$\omega^2 n_{\mathrm{e}1} - \omega_{\mathrm{p}}^2 n_{\mathrm{e}1} - 3k^2 \frac{K_{\mathrm{B}} T_{\mathrm{e}}}{m} n_{\mathrm{e}1} = 0 \tag{5.40}$$

$$\omega^2 = \omega_{\mathrm{p}}^2 + \frac{3}{2} k^2 v_{\mathrm{T}}^2, \tag{5.41}$$

$$n_{\mathrm{e}1} = \overline{n_{\mathrm{e}1}} \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) \tag{5.42}$$

$$\boldsymbol{E}_1 = \overline{\boldsymbol{E}_1} \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta_E). \tag{5.43}$$

$$-\varepsilon_0 \boldsymbol{k} \cdot \overline{\boldsymbol{E}_1} \sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t + \delta_E) = -e\overline{n_{\mathrm{e}1}} \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t). \tag{5.44}$$

$$\sin(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) \cos \delta_E + \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t) \sin \delta_E = \frac{e}{\varepsilon_0} \frac{\overline{n_{\mathrm{e}1}}}{\overline{\boldsymbol{E}_1} \cdot \boldsymbol{k}} \cos(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t). \tag{5.45}$$

$$-\mathrm{i}\omega m \boldsymbol{u}_{\mathrm{e}1} = e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0). \tag{5.46}$$

$$\boldsymbol{u}_{\mathrm{e}1} = -\frac{e}{\omega} \boldsymbol{E}_1, \tag{5.47}$$

$$\boldsymbol{k} \times \boldsymbol{E}_1 = \omega \boldsymbol{B}_1, \tag{5.48}$$

$$m n_0 \frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} = -e n_0 \boldsymbol{E}_1 - 3K_{\mathrm{B}} T_{\mathrm{e}} \nabla n_{\mathrm{e}1}, \tag{5.49}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.50}$$

$$M n_0 \frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = e n_0 \boldsymbol{E}_1 - 3K_{\mathrm{B}} T_{\mathrm{i}} \nabla n_{\mathrm{i}1}, \tag{5.51}$$

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} = 0, \tag{5.52}$$

$$\boldsymbol{E}_1 = -\nabla \phi_1, \tag{5.53}$$

$$\varepsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = e(n_{\mathrm{i}1} - n_{\mathrm{e}1}). \tag{5.54}$$

$$n_{\mathrm{e}1} = n_{\mathrm{e}} - n_{\mathrm{e}0} = n_0 \left(\exp \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}} - 1 \right) \sim n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}} \tag{5.55}$$

$$M n_0 \frac{\partial \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = -e n_0 \triangle \phi_1 - 3K_{\mathrm{B}} T_{\mathrm{i}} \triangle n_{\mathrm{i}1}, \tag{5.56}$$

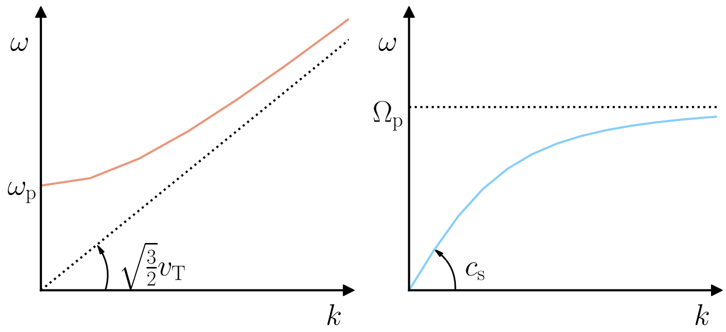
$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla} \cdot \boldsymbol{u}_{\mathrm{i}1} = 0, \tag{5.57}$$

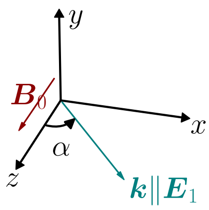
$$-\varepsilon_0 \triangle \phi_1 = e n_{\mathrm{i}1} - e n_0 \frac{e\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.58}$$

$$\omega^2 M n_{\mathrm{i}1} = e n_0 k^2 \phi_1 + 3K_{\mathrm{B}} T_{\mathrm{i}} k^2 n_{\mathrm{i}1}, \tag{5.59}$$

$$\varepsilon_0 k^2 \phi_1 = e n_{\mathrm{i}1} - e^2 n_0 \frac{\phi_1}{K_{\mathrm{B}} T_{\mathrm{e}}}. \tag{5.60}$$

$$\omega^2 = \frac{e^2 n_0 k^2}{M \left(\varepsilon_0 k^2 + \frac{e^2 n_0}{K_{\mathrm{B}} T_{\mathrm{e}}} \right)} + \frac{3K_{\mathrm{B}} T_{\mathrm{i}} k^2}{M}. \tag{5.61}$$





$$\omega^2 = \frac{\Omega_{\mathrm{p}}^2\lambda_{\mathrm{D}}^2k^2}{\lambda_{\mathrm{D}}^2k^2+1} + \frac{3}{2}v_{\mathrm{Ti}}^2k^2. \tag{5.62}$$

$$\omega^2 = k^2\left(\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{M} + 3\frac{K_{\mathrm{B}}T_{\mathrm{i}}}{M}\right). \tag{5.63}$$

5.5 Electrostatic waves in background magnetic field

$$-\mathrm{i}\omega m\boldsymbol{u}_{\mathrm{e}1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1}\times\boldsymbol{B}_0), \tag{5.64}$$
$$-\mathrm{i}\omega n_{\mathrm{e}1} + \mathrm{i}n_0\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.65}$$
$$\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{E}_1 = -\frac{e}{\varepsilon_0}n_{\mathrm{e}1}, \tag{5.66}$$

$$\boldsymbol{E}_1 = \frac{\mathrm{i}\omega m}{e}\boldsymbol{u}_{\mathrm{e}1} - \boldsymbol{u}_{\mathrm{e}1}\times\boldsymbol{B}_0, \tag{5.67}$$
$$\boldsymbol{k}\cdot\boldsymbol{E}_1 = \frac{\mathrm{i}en_0}{\varepsilon_0\omega}\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}1}. \tag{5.68}$$

$$E_1 = \frac{\mathrm{i}en_0}{\varepsilon_0\omega k}\boldsymbol{k}\cdot\boldsymbol{u}_{\mathrm{e}1}. \tag{5.69}$$

$$\begin{pmatrix} E_1 \sin\alpha \\ 0 \\ E_1 \cos\alpha \end{pmatrix} = \frac{\mathrm{i}en_0}{\varepsilon_0 \omega} \left(k u_{\mathrm{e}1,x} \sin\alpha + k u_{\mathrm{e}1,z} \cos\alpha\right) \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} = \tag{5.70}$$

$$= \frac{\mathrm{i}\omega m}{e} \begin{pmatrix} u_{\mathrm{e}1,x} \\ u_{\mathrm{e}1,y} \\ u_{\mathrm{e}1,z} \end{pmatrix} - B_0 \begin{pmatrix} u_{\mathrm{e}1,y} \\ -u_{\mathrm{e}1,x} \\ 0 \end{pmatrix}. \tag{5.71}$$

$$\begin{pmatrix} u_{\mathrm{e}1,x} \\ u_{\mathrm{e}1,y} \\ u_{\mathrm{e}1,z} \end{pmatrix} - \frac{\omega_\mathrm{p}^2}{\omega^2} \left(u_{\mathrm{e}1,x} \sin\alpha + u_{\mathrm{e}1,z} \cos\alpha\right) \begin{pmatrix} \sin\alpha \\ 0 \\ \cos\alpha \end{pmatrix} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega} \begin{pmatrix} u_{\mathrm{e}1,y} \\ -u_{\mathrm{e}1,x} \\ 0 \end{pmatrix} = 0. \tag{5.72}$$

$$M_u \cdot \boldsymbol{u}_{\mathrm{e}1} = 0, \tag{5.73}$$

$$M_u = \begin{bmatrix} 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha \\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega} & 1 & 0 \\ -\frac{\omega_\mathrm{p}^2}{\omega^2}\sin\alpha\cos\alpha & 0 & 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha \end{bmatrix} \tag{5.74}$$

$$\det M_u = -\frac{\omega_\mathrm{p}^4}{\omega^4}\sin^2\alpha\cos^2\alpha + \left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\cos^2\alpha\right)\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\sin^2\alpha - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0\,, \tag{5.75}$$

$$\left(1 - \frac{\omega_\mathrm{p}^2}{\omega^2}\right)\left(1 - \frac{\omega_\mathrm{c}^2}{\omega^2}\right) = 0, \tag{5.76}$$

$$\omega^2 = \omega_\mathrm{p}^2, \quad \text{and} \quad \omega^2 = \omega_\mathrm{c}^2. \tag{5.77}$$

$$1 - \frac{\omega_\mathrm{p}^2}{\omega^2} - \frac{\omega_\mathrm{c}^2}{\omega^2} = 0, \tag{5.78}$$

$$\omega^2 = \omega_\mathrm{p}^2 + \omega_\mathrm{c}^2 \equiv \omega_\mathrm{h}^2. \tag{5.79}$$

$$-\mathrm{i}\omega M \boldsymbol{u}_{\mathrm{i}1} = -e\mathrm{i}\boldsymbol{k}\phi_1 + e\boldsymbol{u}_{\mathrm{i}1} \times \boldsymbol{B}_0 \tag{5.80}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = -e\mathrm{i}k\phi_1 + e u_{\mathrm{i}1,y} B_0, \tag{5.81}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,y} = -e u_{\mathrm{i}1,x} B_0, \tag{5.82}$$

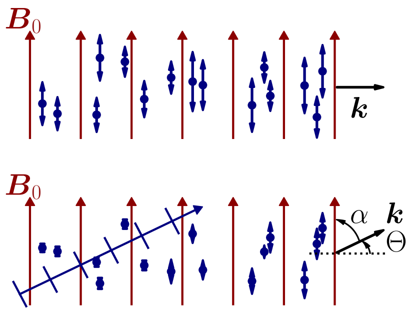
$$u_{\mathrm{i}1,x} = \frac{ek\phi_1}{\omega M}\left(1 - \frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}. \tag{5.83}$$

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.84}$$

$$n_{\mathrm{e}1} = n_0 \frac{e\phi_1}{K_\mathrm{B} T_\mathrm{e}}. \tag{5.85}$$

$$u_{\mathrm{i}1,x} = \frac{\omega e\phi_1}{K_\mathrm{B} T_\mathrm{e} k}, \tag{5.86}$$

$$\omega^2 - \Omega_\mathrm{c}^2 = \frac{K_\mathrm{B} T_\mathrm{e}}{M} k^2. \tag{5.87}$$



$$c_{\mathrm{s}}^2 = \left(\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{M} + \frac{3K_{\mathrm{B}}T_{\mathrm{i}}}{M}\right) \tag{5.88}$$

$$\omega^2 = \Omega_{\mathrm{c}}^2 + k^2 c_{\mathrm{s}}^2. \tag{5.89}$$

$$u_{\mathrm{e}1,x} = -\frac{ek\phi_1}{\omega m}\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right)^{-1}. \tag{5.90}$$

$$n_{\mathrm{i}1} = n_0 \frac{k}{\omega} u_{\mathrm{i}1,x}, \tag{5.91}$$

$$n_{\mathrm{e}1} = n_0 \frac{k}{\omega} u_{\mathrm{e}1,x}. \tag{5.92}$$

$$M\left(1 - \frac{\Omega_{\mathrm{c}}^2}{\omega^2}\right) = -m\left(1 - \frac{\omega_{\mathrm{c}}^2}{\omega^2}\right), \tag{5.93}$$

$$\omega^2(M+m) = m\omega_{\mathrm{c}}^2 + M\Omega_{\mathrm{c}}^2 = e^2B_0^2\left(\frac{1}{m} + \frac{1}{M}\right) = e^2B_0^2\left(\frac{m+M}{mM}\right) \tag{5.94}$$

$$\omega^2 = \frac{eB_0}{m}\frac{eB_0}{M} = \omega_{\mathrm{c}}\Omega_{\mathrm{c}} \equiv \omega_{\mathrm{d}}^2. \tag{5.95}$$

5.6 Electromagnetic waves

$$\boldsymbol{\nabla} \times \boldsymbol{E}_1 = -\frac{\partial \boldsymbol{B}_1}{\partial t}, \tag{5.96}$$

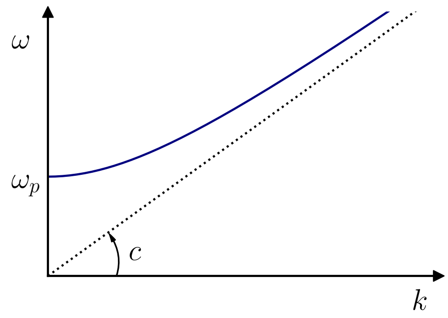
$$\frac{1}{\mu_0}\boldsymbol{\nabla} \times \boldsymbol{B}_1 = \varepsilon_0 \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.97}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0. \tag{5.98}$$

$$\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}_1}{\partial t} = -\frac{\partial^2 \boldsymbol{B}_1}{\partial t^2}, \tag{5.99}$$

$$\frac{1}{\mu_0\varepsilon_0}\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{B}_1 = \boldsymbol{\nabla} \times \frac{\partial \boldsymbol{E}_1}{\partial t}, \tag{5.100}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B}_1 = 0. \tag{5.101}$$



$$-c^2\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{B}_1)=\omega^2\boldsymbol{B}_1, \tag{5.102}$$
$$\boldsymbol{k}\cdot\boldsymbol{B}_1=0, \tag{5.103}$$

$$-c^2\left[(\boldsymbol{k}\cdot\boldsymbol{B}_1)\boldsymbol{k}-k^2\boldsymbol{B}_1\right]=\omega^2\boldsymbol{B}_1, \tag{5.104}$$
$$\boldsymbol{k}\cdot\boldsymbol{B}_1=0. \tag{5.105}$$

$$c^2k^2\boldsymbol{B}_1=\omega^2\boldsymbol{B}_1, \tag{5.106}$$
$$\omega^2=c^2k^2. \tag{5.107}$$

$$\boldsymbol{\nabla}\times\boldsymbol{E}_1=-\frac{\partial\boldsymbol{B}_1}{\partial t}, \tag{5.108}$$
$$\frac{1}{\mu_0}\boldsymbol{\nabla}\times\boldsymbol{B}_1=\varepsilon_0\frac{\partial\boldsymbol{E}_1}{\partial t}+\boldsymbol{j}_1. \tag{5.109}$$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\boldsymbol{E}_1=-\frac{1}{c^2}\frac{\partial^2\boldsymbol{E}_1}{\partial t^2}-\frac{1}{\varepsilon_0c^2}\frac{\partial\boldsymbol{j}_1}{\partial t}. \tag{5.110}$$

$$-(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k}+k^2\boldsymbol{E}_1=\frac{\mathrm{i}\omega}{\varepsilon_0c^2}\boldsymbol{j}_1+\frac{\omega^2}{c^2}\boldsymbol{E}_1. \tag{5.111}$$

$$(\omega^2-c^2k^2)\boldsymbol{E}_1=-\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}_1. \tag{5.112}$$

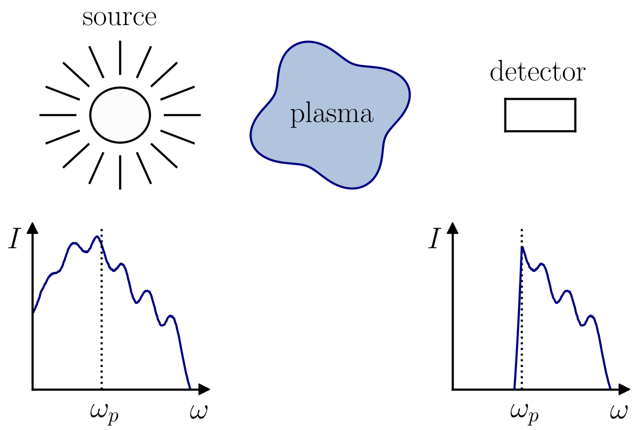
$$\boldsymbol{j}_1=-n_0e\boldsymbol{u}_{\mathrm{e}1}. \tag{5.113}$$

$$m\frac{\partial\boldsymbol{u}_{\mathrm{e}1}}{\partial t}=-e\boldsymbol{E}_1. \tag{5.114}$$

$$\boldsymbol{j}_1=-\frac{n_0e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1, \tag{5.115}$$

$$(\omega^2-c^2k^2)\boldsymbol{E}_1=\frac{\mathrm{i}\omega}{\varepsilon_0}\frac{n_0e^2}{\mathrm{i}\omega m}\boldsymbol{E}_1=\frac{n_0e^2}{\varepsilon_0m}\boldsymbol{E}_1=\omega_{\mathrm{p}}^2\boldsymbol{E}_1. \tag{5.116}$$

$$\omega^2=\omega_{\mathrm{p}}^2+c^2k^2. \tag{5.117}$$


source
plasma
detector
I
ω_p
ω
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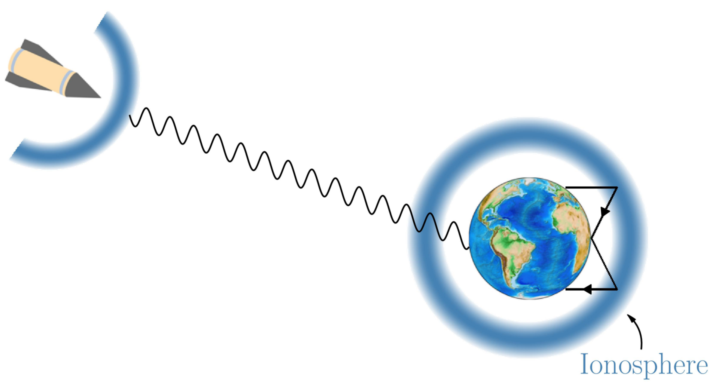
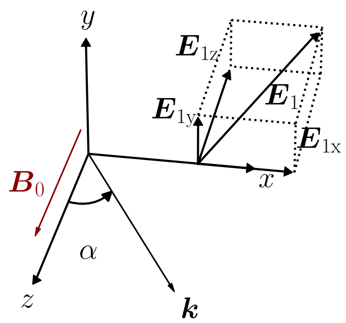
ω_p
ω


Ionosphere



$$-\mathrm{i}\omega m\boldsymbol{u}_{\mathrm{e}1} = -e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{B}_0), \tag{5.118}$$

$$\mathrm{i}\boldsymbol{k} \times \boldsymbol{E}_1 = \mathrm{i}\omega\boldsymbol{B}_1, \tag{5.119}$$

$$\frac{\mathrm{i}}{\mu_0}\boldsymbol{k} \times \boldsymbol{B}_1 = -\varepsilon_0\mathrm{i}\omega\boldsymbol{E}_1 - en_0\boldsymbol{u}_{\mathrm{e}1}. \tag{5.120}$$

$$\boldsymbol{u}_{\mathrm{e}1} = -\mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{eB_0}{m\omega}\boldsymbol{u}_{\mathrm{e}1} \times \boldsymbol{b}_0, \tag{5.121}$$

$$\boldsymbol{u}_{\mathrm{e}1} = -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0\omega}{\omega^2 en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1), \tag{5.122}$$

$$\begin{aligned} -\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1) = \\ -\mathrm{i}\frac{e}{m\omega}\boldsymbol{E}_1 - \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}\left[-\mathrm{i}\frac{\varepsilon_0\omega}{en_0}\boldsymbol{E}_1 - \mathrm{i}\frac{c^2\varepsilon_0}{\omega en_0}\boldsymbol{k} \times (\boldsymbol{k} \times \boldsymbol{E}_1)\right] \times \boldsymbol{b}_0. \end{aligned} \tag{5.123}$$

$$\begin{aligned} \omega^2\boldsymbol{E}_1 + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} - c^2k^2\boldsymbol{E}_1 = \\ = \frac{e^2n_0}{\varepsilon_0 m}\boldsymbol{E}_1 - \mathrm{i}\omega_\mathrm{c}\omega\boldsymbol{E}_1 \times \boldsymbol{b}_0 - \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k} \times \boldsymbol{b}_0) + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2k^2(\boldsymbol{E}_1 \times \boldsymbol{b}_0). \end{aligned} \tag{5.124}$$

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\boldsymbol{E}_1 + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2 - c^2k^2)(\boldsymbol{E}_1 \times \boldsymbol{b}_0) + c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)\boldsymbol{k} + \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2(\boldsymbol{k}\cdot\boldsymbol{E}_1)(\boldsymbol{k} \times \boldsymbol{b}_0) = 0. \tag{5.125}$$

$$(\omega^2-\omega_\mathrm{p}^2-c^2k^2)\begin{pmatrix}E_{1x}\\E_{1y}\\E_{1z}\end{pmatrix}+\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2)\begin{pmatrix}E_{1y}\\-E_{1x}\\0\end{pmatrix}+$$
$$+c^2(kE_{1x}\sin\alpha+kE_{1z}\cos\alpha)\begin{pmatrix}k\sin\alpha\\0\\k\cos\alpha\end{pmatrix}+$$
$$+\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2(E_{1x}k\sin\alpha+kE_{1z}\cos\alpha)\begin{pmatrix}0\\-k\sin\alpha\\0\end{pmatrix}=0. \tag{5.126}$$

$$M_{\boldsymbol{E}_1}\cdot\boldsymbol{E}_1=0 \tag{5.127}$$

$$M_{\boldsymbol{E}_1}=\begin{pmatrix}\omega^2-\omega_\mathrm{p}^2-c^2k^2\cos^2\alpha & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2) & c^2k^2\sin\alpha\cos\alpha\\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2\cos^2\alpha) & \omega^2-\omega_\mathrm{p}^2-c^2k^2 & -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}c^2k^2\cos\alpha\sin\alpha\\ c^2k^2\sin\alpha\cos\alpha & 0 & \omega^2-\omega_\mathrm{p}^2-c^2k^2\sin^2\alpha\end{pmatrix}. \tag{5.128}$$

$$\det M_{\boldsymbol{E}_1}=c^4k^4\sin^2\alpha\cos^2\alpha\left[\frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2-c^2k^2)-(\omega^2-\omega_\mathrm{p}^2-c^2k^2)\right]+$$
$$+(\omega^2-\omega_\mathrm{p}^2-c^2k^2\sin^2\alpha)\Big[(\omega^2-\omega_\mathrm{p}^2-c^2k^2\cos^2\alpha)(\omega^2-\omega_\mathrm{p}^2-c^2k^2)-$$
$$-\frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2-c^2k^2)(\omega^2-c^2k^2\cos^2\alpha)\Big]=0. \tag{5.129}$$

$$(\omega^2-\omega_\mathrm{p}^2)\left[(\omega^2-\omega_\mathrm{p}^2-c^2k^2)^2-\frac{\omega_\mathrm{c}^2}{\omega^2}(\omega^2-c^2k^2)^2\right]=0. \tag{5.130}$$

$$\omega^2=\omega_\mathrm{p}^2, \tag{5.131}$$

$$\omega^2-\omega_\mathrm{p}^2-c^2k^2=\pm\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2) \tag{5.132}$$

$$\omega^2-c^2k^2=\frac{\omega_\mathrm{p}^2}{1-\frac{\omega_\mathrm{c}}{\omega}} \tag{5.133}$$

$$\omega^2-c^2k^2=\frac{\omega_\mathrm{p}^2}{1+\frac{\omega_\mathrm{c}}{\omega}}. \tag{5.134}$$

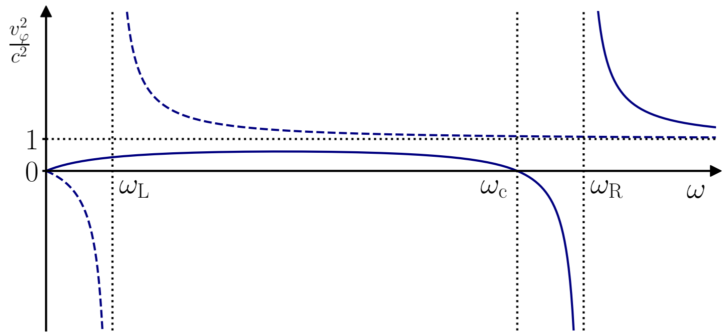
$$\begin{pmatrix}\omega^2-\omega_\mathrm{p}^2-c^2k^2 & \mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2) & 0\\ -\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2) & \omega^2-\omega_\mathrm{p}^2-c^2k^2 & 0\\ 0 & 0 & \omega^2-\omega_\mathrm{p}^2\end{pmatrix}\cdot\begin{pmatrix}E_{1x}\\E_{1y}\\E_{1z}\end{pmatrix}=0, \tag{5.135}$$

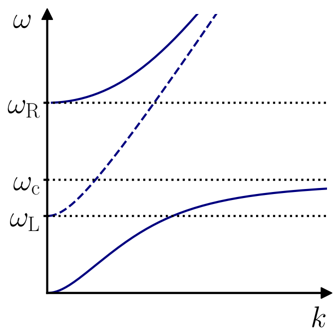
$$(\omega^2-\omega_\mathrm{p}^2-c^2k^2)E_{1x}+\mathrm{i}\frac{\omega_\mathrm{c}}{\omega}(\omega^2-c^2k^2)E_{1y}=0. \tag{5.136}$$

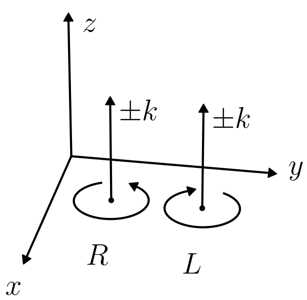
$$E_{1y}=\pm\mathrm{i}E_{1x}. \tag{5.137}$$

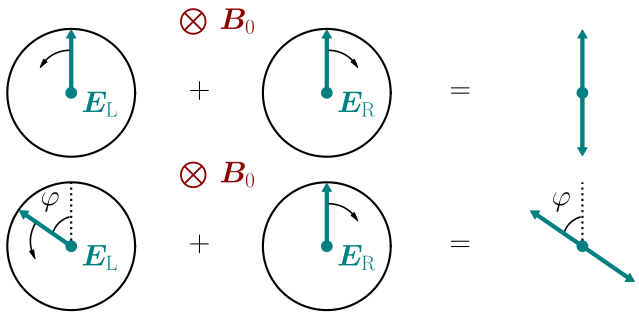
$$\boldsymbol{E}_1=\overline{\boldsymbol{E}_1}\exp\left[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\right]=\overline{\boldsymbol{E}_1}\left[\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)+\mathrm{i}\sin(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\right]. \tag{5.138}$$

$$E_x=\Re(\boldsymbol{E}_1)=\overline{\boldsymbol{E}_1}\cos(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t) \tag{5.139}$$


$\frac{v_\varphi^2}{c^2}$
1
0
ω_L
ω_c
ω_R
ω


ω
ω_R
ω_c
ω_L
k


z
$\pm k$
$\pm k$
y
R
L
x



$$E_y = \Im(\boldsymbol{E}_1) = \overline{\boldsymbol{E}_1}\sin\left(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \delta_E\right), \tag{5.140}$$

$$\boldsymbol{E}_\mathrm{R} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_\mathrm{R}\cdot\boldsymbol{r} - \omega t)]\left[\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y\right], \tag{5.141}$$

$$\boldsymbol{E}_\mathrm{L} = E_0 \exp[\mathrm{i}(\boldsymbol{k}_\mathrm{L}\cdot\boldsymbol{r} - \omega t)]\left[\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y\right]. \tag{5.142}$$

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{E}_\mathrm{R} + \boldsymbol{E}_\mathrm{L}) = \frac{1}{2}E_0\mathrm{e}^{-\mathrm{i}\omega t}\left[\mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{R}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y) + \mathrm{e}^{\mathrm{i}\boldsymbol{k}_\mathrm{L}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)\right]. \tag{5.143}$$

$$\boldsymbol{E} = \frac{1}{2}E_0\mathrm{e}^{-\mathrm{i}\omega t}\left[2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{e}_x\right] = E_0\exp[\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)]\boldsymbol{e}_x, \tag{5.144}$$

$$k_\mathrm{R,L} = \frac{\omega}{c}\sqrt{1 - \frac{\omega_\mathrm{p}^2/\omega^2}{1 \mp \omega_\mathrm{c}/\omega}}. \tag{5.145}$$

$$k_\mathrm{R,L} \sim \frac{\omega}{c}\left(1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\frac{1}{1 \mp \omega_\mathrm{c}/\omega}\right) \sim \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 \pm \frac{\omega_\mathrm{c}}{\omega}\right)\right] \tag{5.146}$$

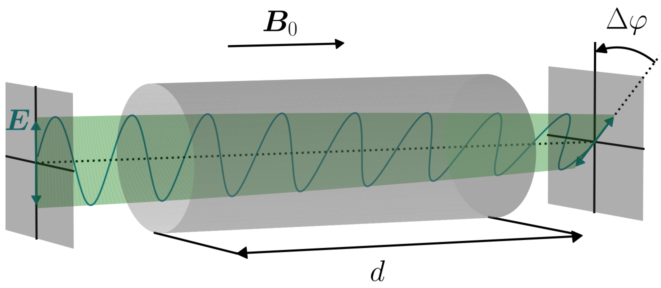
$$k_\mathrm{R,L} = k \pm \Delta k, \tag{5.147}$$

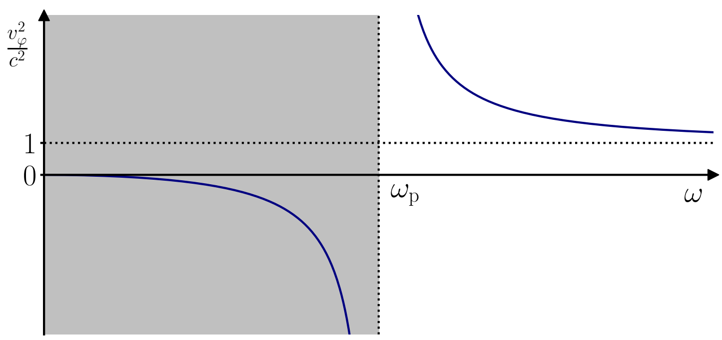
$$k = \frac{\omega}{c}\left[1 - \frac{1}{2}\frac{\omega_\mathrm{p}^2}{\omega^2}\right] \tag{5.148}$$

$$\Delta k = -\frac{1}{2}\frac{\omega_\mathrm{p}^2\omega_\mathrm{c}}{c\omega^2}. \tag{5.149}$$

$$\begin{aligned}\boldsymbol{E} &= \frac{1}{2}E_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\left[\mathrm{e}^{\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x + \mathrm{i}\boldsymbol{e}_y) + \mathrm{e}^{-\mathrm{i}\Delta\boldsymbol{k}\cdot\boldsymbol{r}}(\boldsymbol{e}_x - \mathrm{i}\boldsymbol{e}_y)\right] \\ &= E_0\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)}\left[\cos(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_x + \sin(-\Delta\boldsymbol{k}\cdot\boldsymbol{r})\boldsymbol{e}_y\right].\end{aligned} \tag{5.150}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}z} = -\Delta k. \tag{5.151}$$





$$\varphi = \varphi_0 + \int_0^d \frac{\mathrm{d}\varphi}{\mathrm{d}z}\mathrm{d}z = \varphi_0 + \int_0^d -\Delta k \mathrm{d}z = \varphi_0 + \frac{1}{2c\omega^2}\int_0^d \omega_\mathrm{p}^2(z)\omega_\mathrm{c}(z)\mathrm{d}z =$$

$$= \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c\omega^2}\int_0^d n_0(z)B_0(z)\mathrm{d}z. \tag{5.152}$$

$$\varphi = \varphi_0 + \frac{e^3}{2m^2\varepsilon_0 c}\frac{1}{\omega^2}n_0 B_0 d. \tag{5.153}$$

$$(\omega^2 - \omega_\mathrm{p}^2 - c^2k^2)\left[(\omega^2-\omega_\mathrm{p}^2)(\omega^2-\omega_\mathrm{p}^2-c^2k^2) - \omega_\mathrm{c}^2(\omega^2-c^2k^2)\right] = 0. \tag{5.154}$$

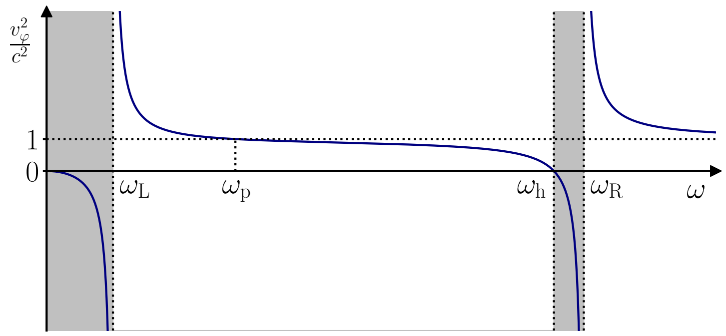
$$\omega^2 = \omega_\mathrm{p}^2 + c^2k^2, \tag{5.155}$$

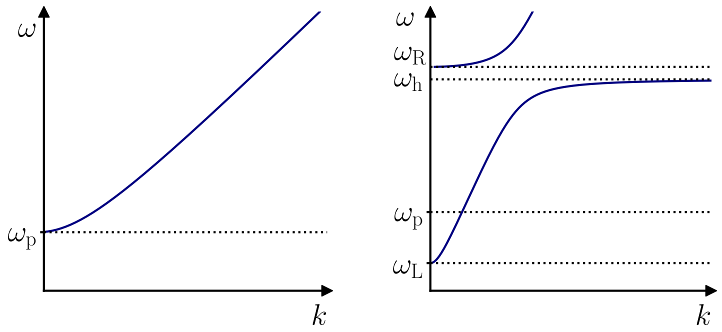
$$(\omega^2-\omega_\mathrm{p}^2)(\omega^2-\omega_\mathrm{p}^2-c^2k^2) = \omega_\mathrm{c}^2(\omega^2-c^2k^2), \tag{5.156}$$

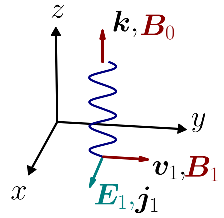
$$\omega^2 - c^2k^2 = \omega_\mathrm{p}^2\frac{\omega^2-\omega_\mathrm{p}^2}{\omega^2-\omega_\mathrm{h}^2}, \tag{5.157}$$

5.7 MHD waves

$$(\omega^2 - c^2k^2)\boldsymbol{E}_1 = -\frac{\mathrm{i}\omega}{\varepsilon_0}\boldsymbol{j}_1. \tag{5.158}$$


$\frac{v_\phi^2}{c^2}$
1
0
ω_L
ω_p
ω_h
ω_R
ω


ω
ω_p
k
ω
ω_R
ω_h
ω_p
ω_L
k


z
$\boldsymbol{k}, \boldsymbol{B}_0$
y
x
$\boldsymbol{v}_1, \boldsymbol{B}_1$
$\boldsymbol{E}_1, \boldsymbol{j}_1$

$$\boldsymbol{j}_1 = j_1\boldsymbol{e}_x = n_0 e(u_{\mathrm{i}1,x} - u_{\mathrm{e}1,x})\boldsymbol{e}_x. \tag{5.159}$$

$$M\frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = e(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{i}1}\times\boldsymbol{B}_0), \tag{5.160}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eu_{\mathrm{i}1,y}B_0, \tag{5.161}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,y} = -eu_{\mathrm{i}1,x}B_0. \tag{5.162}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1,x} = eE_1 + eB_0\frac{eB_0}{\mathrm{i}\omega M}u_{\mathrm{i}1,x} \tag{5.163}$$

$$u_{\mathrm{i}1,x} = \frac{\mathrm{i}e}{\omega M}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1. \tag{5.164}$$

$$u_{\mathrm{i}1,y} = \frac{e}{\omega M}\frac{\Omega_\mathrm{c}}{\omega}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1. \tag{5.165}$$

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m}\left(1-\frac{\omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1, \tag{5.166}$$

$$u_{\mathrm{e}1,y} = -\frac{e}{\omega m}\frac{-\omega_\mathrm{c}}{\omega}\left(1-\frac{\omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1. \tag{5.167}$$

$$u_{\mathrm{e}1,x} = -\frac{\mathrm{i}e}{\omega m}\left(\frac{\omega^2-\omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 \sim \frac{\mathrm{i}e}{\omega m}\frac{\omega^2}{\omega_\mathrm{c}^2}E_1 \to 0 \tag{5.168}$$

$$u_{\mathrm{e}1,y} = \frac{e}{\omega m}\frac{\omega_\mathrm{c}}{\omega}\left(\frac{\omega^2-\omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 \sim -\frac{e}{\omega m}\frac{\omega_\mathrm{c}}{\omega}\frac{\omega^2}{\omega_\mathrm{c}^2}E_1 = -\frac{e}{\omega_\mathrm{c} m}E_1 = -\frac{E_1}{B_0}. \tag{5.169}$$

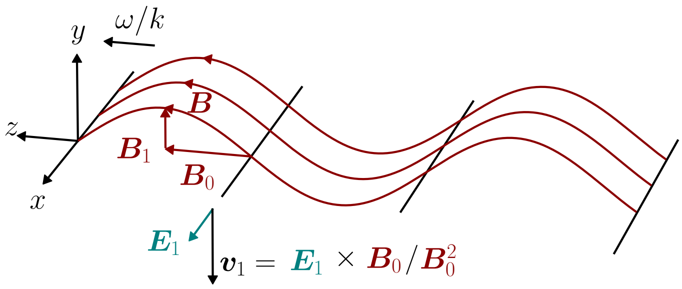
$$\begin{aligned}(\omega^2-c^2k^2)E_1 &= -\frac{\mathrm{i}\omega}{\varepsilon_0}n_0 e\left[\frac{\mathrm{i}e}{\omega M}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 - 0\right] = \\ &= \frac{n_0e^2}{\varepsilon_0 M}\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1 = \Omega_\mathrm{p}^2\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}E_1.\end{aligned} \tag{5.170}$$

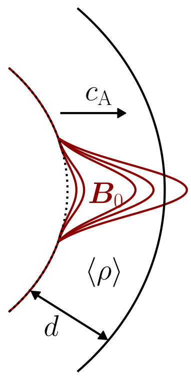
$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2\left(1-\frac{\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1}. \tag{5.171}$$

$$\omega^2 - c^2k^2 = \Omega_\mathrm{p}^2\left(\frac{\omega^2-\Omega_\mathrm{c}^2}{\omega^2}\right)^{-1} \sim -\omega^2\frac{\Omega_\mathrm{p}^2}{\Omega_\mathrm{c}^2} = -\omega^2\frac{n_0M}{\varepsilon_0B_0^2} = -\omega^2\frac{\rho_0}{\varepsilon_0B_0^2}. \tag{5.172}$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{1+\frac{\rho_0}{\varepsilon_0B_0^2}}. \tag{5.173}$$

$$\frac{\omega^2}{k^2} = \frac{c^2}{\frac{\rho\mu_0c^2}{B_0^2}} = \frac{B_0^2}{\mu_0\rho_0}. \tag{5.174}$$





$$\frac{\omega}{k} = \frac{B_0}{\sqrt{\mu_0\rho_0}} \equiv c_{\mathrm{A}}. \tag{5.175}$$

$$\frac{\partial\rho}{\partial t} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{5.176}$$

$$\rho\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nabla p + (\boldsymbol{\nabla}\times\boldsymbol{B})\times\boldsymbol{B}/\mu_0, \tag{5.177}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho^\gamma}\right) = 0, \tag{5.178}$$

$$\frac{\partial\boldsymbol{B}}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{5.179}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0. \tag{5.180}$$

$$\frac{\partial\rho_1}{\partial t} + \rho_0(\boldsymbol{\nabla}\cdot\boldsymbol{u}_1) = 0, \tag{5.181}$$

$$\rho_0\frac{\partial\boldsymbol{u}_1}{\partial t} = -\nabla p_1 + (\boldsymbol{\nabla}\times\boldsymbol{B}_1)\times\boldsymbol{B}_0/\mu_0, \tag{5.182}$$

$$\frac{\partial\boldsymbol{B}_1}{\partial t} = \boldsymbol{\nabla}\times(\boldsymbol{u}_1\times\boldsymbol{B}_0), \tag{5.183}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{B}_1 = 0. \tag{5.184}$$

$$
\begin{aligned}
0 &= \left\{\frac{\partial}{\partial t} + \boldsymbol{u}_1 \cdot \boldsymbol{\nabla}\right\} \left[\frac{p_0 + p_1}{(\rho_0 + \rho_1)^\gamma}\right] = \\
&= \frac{1}{(\rho_0 + \rho_1)^\gamma} \left[\frac{\partial p_1}{\partial t} + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}) p_0\right] + (p_0 + p_1) \left\{\frac{-\gamma}{(\rho_0 + \rho_1)^{\gamma+1}} \left[\frac{\partial \rho_1}{\partial t} + (\boldsymbol{u}_1 \cdot \boldsymbol{\nabla}) \rho_0\right]\right\}.
\end{aligned} \tag{5.185}
$$

$$
0 = \frac{1}{(\rho_0 + \rho_1)^\gamma} \left[\frac{\partial p_1}{\partial t} - \gamma \frac{p_0 + p_1}{\rho_0 + \rho_1} \frac{\partial \rho_1}{\partial t}\right] = \frac{1}{(\rho_0 + \rho_1)^\gamma} \left\{\frac{\partial p_1}{\partial t} - c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t}\right\}, \tag{5.186}
$$

$$
\frac{\partial p_1}{\partial t} - c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t} = 0. \tag{5.187}
$$

$$
\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0} \nabla \frac{\partial p_1}{\partial t} + (\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}_1}{\partial t}) \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0}. \tag{5.188}
$$

$$
\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = -\frac{1}{\rho_0} \nabla \left(c_\mathrm{s}^2 \frac{\partial \rho_1}{\partial t}\right) + (\boldsymbol{\nabla} \times \frac{\partial \boldsymbol{B}_1}{\partial t}) \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0} \tag{5.189}
$$

$$
\frac{\partial^2 \boldsymbol{u}_1}{\partial t^2} = c_\mathrm{s}^2 \nabla \left(\nabla \cdot \boldsymbol{u}_1\right) + \{\boldsymbol{\nabla} \times [\boldsymbol{\nabla} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0)]\} \times \frac{\boldsymbol{B}_0}{\rho_0 \mu_0}. \tag{5.190}
$$

$$
\begin{aligned}
\omega^2 \boldsymbol{u}_1 &= c_\mathrm{s}^2 \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{u}_1) + \{\boldsymbol{k} \times [\boldsymbol{k} \times (\boldsymbol{u}_1 \times \boldsymbol{B}_0)]\} \times \frac{\boldsymbol{B}_0}{\mu_0 \rho_0} \\
&= c_\mathrm{s}^2 \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{u}_1) + \{\boldsymbol{k} \times [\boldsymbol{k} \times (\boldsymbol{u}_1 \times \boldsymbol{b}_0)]\} \times \boldsymbol{b}_0 c_\mathrm{A}^2,
\end{aligned} \tag{5.191}
$$

$$
\begin{aligned}
\{\boldsymbol{k} \times [\boldsymbol{k} \times (\boldsymbol{u}_1 \times \boldsymbol{b}_0)]\} \times \boldsymbol{b}_0 &= \{\boldsymbol{k} \times [\boldsymbol{u}_1(\boldsymbol{k} \cdot \boldsymbol{b}_0) - \boldsymbol{b}_0(\boldsymbol{k} \cdot \boldsymbol{u}_1)]\} \times \boldsymbol{b}_0 = \\
&= (\boldsymbol{k} \cdot \boldsymbol{b}_0) \left[(\boldsymbol{k} \times \boldsymbol{u}_1) \times \boldsymbol{b}_0\right] - (\boldsymbol{k} \cdot \boldsymbol{u}_1) \left[(\boldsymbol{k} \times \boldsymbol{b}_0) \times \boldsymbol{b}_0\right] = \\
&= -(\boldsymbol{k} \cdot \boldsymbol{b}_0) \left[\boldsymbol{k}(\boldsymbol{u}_1 \cdot \boldsymbol{b}_0) - \boldsymbol{u}_1(\boldsymbol{k} \cdot \boldsymbol{b}_0)\right] + (\boldsymbol{k} \cdot \boldsymbol{u}_1) \left[\boldsymbol{k} b_0^2 - \boldsymbol{b}_0(\boldsymbol{k} \cdot \boldsymbol{b}_0)\right] = \\
&= (\boldsymbol{k} \cdot \boldsymbol{b}_0)^2 \boldsymbol{u}_1 - (\boldsymbol{k} \cdot \boldsymbol{u}_1)(\boldsymbol{k} \cdot \boldsymbol{b}_0) \boldsymbol{b}_0 + \boldsymbol{k} \left[\boldsymbol{k} \cdot \boldsymbol{u}_1 - (\boldsymbol{k} \cdot \boldsymbol{b}_0)(\boldsymbol{u}_1 \cdot \boldsymbol{b}_0)\right].
\end{aligned} \tag{5.192}
$$

$$
\omega^2 \boldsymbol{u}_1 = c_\mathrm{s}^2 \boldsymbol{k}(\boldsymbol{k} \cdot \boldsymbol{u}_1) + c_\mathrm{A}^2 \left\{(\boldsymbol{k} \cdot \boldsymbol{b}_0)^2 \boldsymbol{u}_1 - (\boldsymbol{k} \cdot \boldsymbol{u}_1)(\boldsymbol{k} \cdot \boldsymbol{b}_0) \boldsymbol{b}_0 + \boldsymbol{k} \left[\boldsymbol{k} \cdot \boldsymbol{u}_1 - (\boldsymbol{k} \cdot \boldsymbol{b}_0)(\boldsymbol{u}_1 \cdot \boldsymbol{b}_0)\right]\right\}, \tag{5.193}
$$

$$
\frac{\omega^2 \boldsymbol{u}_1}{c_\mathrm{A}^2} = k^2 \cos^2 \alpha \boldsymbol{u}_1 - (\boldsymbol{k} \cdot \boldsymbol{u}_1) k \cos \alpha \boldsymbol{b}_0 + \boldsymbol{k} \left[\left(1 + \frac{c_\mathrm{s}^2}{c_\mathrm{A}^2}\right) (\boldsymbol{k} \cdot \boldsymbol{u}_1) - k \cos \alpha (\boldsymbol{u}_1 \cdot \boldsymbol{b}_0)\right]. \tag{5.194}
$$

$$
\begin{aligned}
\omega^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) &= k^2 c_\mathrm{A}^2 \cos^2 \alpha (\boldsymbol{u}_1 \cdot \boldsymbol{k}) - c_\mathrm{A}^2 k^2 \cos^2 \alpha (\boldsymbol{k} \cdot \boldsymbol{u}_1) + \\
&+ c_\mathrm{A}^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) k^2 + c_\mathrm{s}^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) k^2 - c_\mathrm{A}^2 k^3 \cos \alpha (\boldsymbol{u}_1 \cdot \boldsymbol{b}_0).
\end{aligned} \tag{5.195}
$$

$$
\left[-\omega^2 + c_\mathrm{A}^2 k^2 + c_\mathrm{s}^2 k^2\right] (\boldsymbol{u}_1 \cdot \boldsymbol{k}) = c_\mathrm{A}^2 k^3 \cos \alpha (\boldsymbol{b}_0 \cdot \boldsymbol{u}_1). \tag{5.196}
$$

$$
\begin{aligned}
\omega^2 (\boldsymbol{b}_0 \cdot \boldsymbol{u}_1) &= c_\mathrm{A}^2 k^2 \cos^2 \alpha (\boldsymbol{u}_1 \cdot \boldsymbol{b}_0) - c_\mathrm{A}^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) k \cos \alpha + \\
&+ c_\mathrm{A}^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) k \cos \alpha + c_\mathrm{s}^2 (\boldsymbol{k} \cdot \boldsymbol{u}_1) k \cos \alpha - c_\mathrm{A}^2 k^2 \cos^2 \alpha (\boldsymbol{u}_1 \cdot \boldsymbol{b}_0),
\end{aligned} \tag{5.197}
$$

$$\omega^2(\boldsymbol{b}_0 \cdot \boldsymbol{u}_1) = c_\mathrm{s}^2(\boldsymbol{k} \cdot \boldsymbol{u}_1)k\cos\alpha. \tag{5.198}$$

$$\frac{\boldsymbol{b}_0 \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{k}} = \frac{-\omega^2 + c_\mathrm{A}^2 k^2 + c_\mathrm{s}^2 k^2}{c_\mathrm{A}^2 k^3 \cos\alpha} \tag{5.199}$$

$$\frac{\boldsymbol{b}_0 \cdot \boldsymbol{u}_1}{\boldsymbol{u}_1 \cdot \boldsymbol{k}} = \frac{c_\mathrm{s}^2 k\cos\alpha}{\omega^2}. \tag{5.200}$$

$$\frac{-\omega^2 + c_\mathrm{A}^2 k^2 + c_\mathrm{s}^2 k^2}{c_\mathrm{A}^2 k^3 \cos\alpha} = \frac{c_\mathrm{s}^2 k\cos\alpha}{\omega^2} \tag{5.201}$$

$$\omega^4 - \omega^2 k^2(c_\mathrm{A}^2 + c_\mathrm{s}^2) + c_\mathrm{s}^2 c_\mathrm{A}^2 k^4 \cos^2\alpha = 0. \tag{5.202}$$

$$\left(\frac{\omega}{k}\right)^2 = \left[\frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 4c_\mathrm{s}^2 c_\mathrm{A}^2\cos^2\alpha + 2c_\mathrm{A}^2 c_\mathrm{s}^2}\right]. \tag{5.203}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 - 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}|c_\mathrm{s}^2 - c_\mathrm{A}^2| = \left\{ \begin{array}{c} c_\mathrm{s}^2 \\ c_\mathrm{A}^2 \end{array} \right. , \tag{5.204}$$

$$\left(\frac{\omega}{k}\right)^2 = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}\sqrt{c_\mathrm{s}^4 + c_\mathrm{A}^4 + 2c_\mathrm{A}^2 c_\mathrm{s}^2} = \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) \pm \frac{1}{2}(c_\mathrm{s}^2 + c_\mathrm{A}^2) = \left\{ \begin{array}{c} c_\mathrm{s}^2 + c_\mathrm{A}^2 \\ 0 \end{array} \right. , \tag{5.205}$$

$$\left(\frac{\omega}{k}\right)^2 = \left\{ \begin{array}{c} 0 \\ c_\mathrm{s}^2 \end{array} \right. , \tag{5.206}$$

Chapter 6

Diffusion

6.1 Diffusion in weakly ionised plasmas

In Section **??** we introduced an additional term to the Euler equation accounting for collisions between particles. We defined the particle flux Γ

$$\boldsymbol{\Gamma} = n\boldsymbol{u}, \tag{6.1}$$

where n was the particle density and $\boldsymbol{u}$ their speed.

The formalism used here is similar. For simplicity, let us assume that the particles flow along the x-axis of the Cartesian coordinate system. Due to collisions occurring in the spatial $\mathrm{d}x$, a fraction of the flux $\mathrm{d}\Gamma$ is lost (see Fig. 6.1). We may write

$$\mathrm{d}\Gamma = -n_\mathrm{n}\sigma\Gamma\mathrm{d}x, \tag{6.2}$$

where n_n is the density of neutral particles because we expect the collisions to be dominated by neutral particles. σ is the effective cross-section of the collisions. From the effective cross-section, we may define useful quantities $\lambda_s = 1/n_\mathrm{n}\sigma$ to be the mean free path between consecutive collisions and similarly the mean collisional frequency

$$\nu = n_\mathrm{n}\langle\sigma v\rangle \approx n_\mathrm{n}\sigma\boldsymbol{u}, \tag{6.3}$$

where the averaging is performed over the velocity space. The averaging is performed because quantum mechanics teaches us that the effective cross-section may be a function of velocity. The mean period between consecutive collisions is then $\tau = 1/\nu$.

We take the time derivative of (6.1) and (6.2), where for simplicity we neglect the possible time dependence of all variables apart from the coordinates and velocity. Then using the definition of Γ we have

$$\frac{\mathrm{d}\boldsymbol{\Gamma}}{\mathrm{d}t} = n\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -n_\mathrm{n}\sigma n\boldsymbol{u}\frac{\mathrm{d}x}{\mathrm{d}t} = -n_\mathrm{n}\sigma nu\boldsymbol{u} = -n\nu\boldsymbol{u}, \tag{6.4}$$

Hence

$$\frac{\mathrm{d}\boldsymbol{u}}{\mathrm{d}t} = -\nu\boldsymbol{u}, \tag{6.5}$$

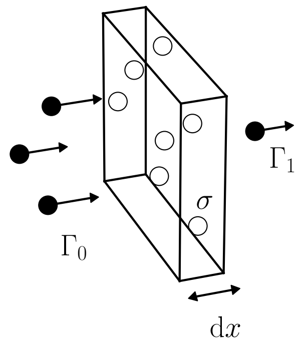


Figure 6.1: Illustration of the model of the change of the particle flux via collisions.

which is the (average) velocity change due to the collisions.

Hence the Euler equation with the magnetic field set to zero and with the collisional term added reads

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = qn\boldsymbol{E} - \nabla p - mn\nu\boldsymbol{u} \tag{6.6}$$

Let's investigate a system where $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u} = 0$ (in such a system we only find a constant motion due to the diffusion). Then

$$\boldsymbol{u} = \frac{q\boldsymbol{E}}{m\nu} - \frac{K_\mathrm{B}T\nabla n}{mn\nu} = \frac{\boldsymbol{\Gamma}}{n}, \tag{6.7}$$

where we used the definition of the particle flux. By introducing the mobility $\mu = \frac{q \operatorname{sgn} q}{m\nu}$ and the diffusion coefficient $D = \frac{K_\mathrm{B}T}{m\nu}$ we have for the particle species j

$$\boldsymbol{\Gamma}_j = \operatorname{sgn} q_j\, n\mu_j\boldsymbol{E} - D_j\nabla n. \tag{6.8}$$

For $\boldsymbol{E} = 0$ we have

$$\boldsymbol{\Gamma} = -D\nabla n, \tag{6.9}$$

which is a form of *Fick's law* for diffusion. A simple interpretation of this useful relation is that the particles flow from locations with a higher density to lower density regions (as illustrated in Fig. 6.2), which is to be expected. In plasmas, Fick's law is generalised to (6.8) noting the effect of the electric field. One should also note that the coefficients μ_j and D_j may generally be different for various particle species (most often electrons and ions) and hence the diffusion in plasmas may cause a separation of charges and thus drive an electric field. However, to fulfill the quasineutrality of plasma, $\boldsymbol{\Gamma}_\mathrm{i} = \boldsymbol{\Gamma}_\mathrm{e} = \boldsymbol{\Gamma}$ on scales $L \gg \lambda_\mathrm{D}$, which allows us to derive the necessary electric field that will keep the plasma quasineutral.

By assuming the plasma approximation $n_\mathrm{e} = n_\mathrm{i} = n$ we have

$$n\mu_\mathrm{i}\boldsymbol{E} - D_\mathrm{i}\nabla n = -n\mu_\mathrm{e}\boldsymbol{E} - D_\mathrm{e}\nabla n \tag{6.10}$$

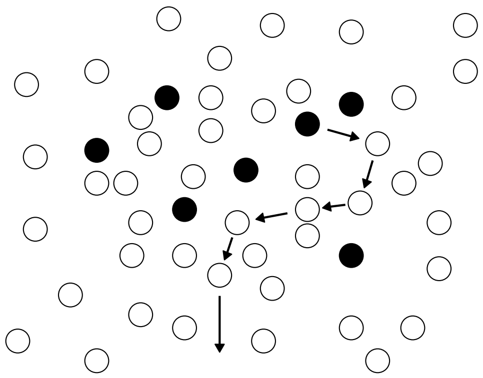


Figure 6.2: Illustrative concept of diffusion. An overabundance of the black "particles" is decreased in time via collisions with white "particles" and their local density thus decreases.

and hence

$$\boldsymbol{E} = \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} \tag{6.11}$$

is the induced electric field triggered by the separation of charges. This is a repulsive electric field, which stops the charges from further separation due to diffusion and hence keeps plasma neutral on large scales. This electric field causes the electrons to slow down and accelerate ions simultaneously, so that the bulk of plasma (with charges separated on scales smaller than λ_D) diffuses according to modified Fick's law. This is the basic picture of *ambipolar diffusion.*

Then

$$\begin{aligned}
\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_\mathrm{i} &= n\mu_\mathrm{i} \frac{D_\mathrm{i} - D_\mathrm{e}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \frac{\nabla n}{n} - D_\mathrm{i} \nabla n = \\
&= \frac{\mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{i} D_\mathrm{e} - \mu_\mathrm{i} D_\mathrm{i} - \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = - \frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \nabla n = \\
&\equiv -D_\mathrm{a} \nabla n,
\end{aligned} \tag{6.12}$$

where D_a is the coefficient of the ambipolar diffusion. For $\mu_\mathrm{e} \gg \mu_\mathrm{i}$ and using the definition of the mobility coefficient we have

$$D_\mathrm{a} = \frac{\mu_\mathrm{i} D_\mathrm{e} + \mu_\mathrm{e} D_\mathrm{i}}{\mu_\mathrm{i} + \mu_\mathrm{e}} \approx D_\mathrm{i} + \frac{\mu_\mathrm{i}}{\mu_\mathrm{e}} D_\mathrm{e} = \left(1 + \frac{T_\mathrm{e}}{T_\mathrm{i}}\right) D_\mathrm{i}. \tag{6.13}$$

Thus the coefficient of ambipolar diffusion is mostly determined by the diffusion coefficients for ions with some correction.

Typical time evolution by diffusion

As an example, let's take the continuum equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot n\boldsymbol{u} = 0, \tag{6.14}$$

and hence

$$\frac{\partial n}{\partial t} = -\boldsymbol{\nabla} \cdot n\boldsymbol{u} = -\boldsymbol{\nabla} \cdot \boldsymbol{\Gamma} = D_\mathrm{a} \triangle n. \tag{6.15}$$

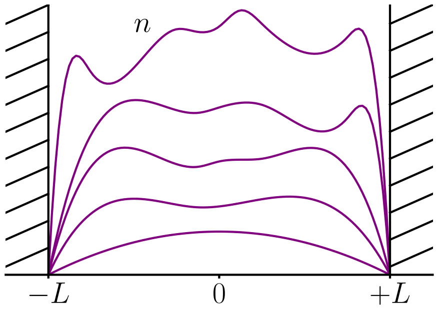


Figure 6.3: A model of plasma diffusion confined between two infinite plane-parallel walls. A series of plasma density snapshots is represented by the purple lines. At the beginning, the density profile is quite complex. Due to diffusion, it simplifies with time, large spatial frequencies (small-scale features) decay the fastest.

This equation is similar to the heat conduction equation. It may be solved by assuming the 1-D solution by separating the variables $n(x,t) = T(t)S(x)$, where T and S are some functions. Then

$$S\frac{\mathrm{d}T}{\mathrm{d}t} = D_{\mathrm{a}}T\,\triangle S, \tag{6.16}$$

and hence

$$\frac{1}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{D_{\mathrm{a}}}{S}\triangle S = \mathrm{const} = -\frac{1}{\tau}. \tag{6.17}$$

Both sides of the equations must equal the same constant because the left-hand side depends purely on time; thus, the right-hand side can't depend on time and vice versa for the spatial dependency of the right-hand side. For dimension reasons, this constant must have the dimension of reciprocal time. The equation may then be solved by parts

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{T}{\tau} \quad \rightarrow \quad T = T_0\mathrm{e}^{-\frac{t}{\tau}} \tag{6.18}$$

and

$$\frac{\mathrm{d}^2S}{\mathrm{d}x^2} = -\frac{1}{D_{\mathrm{a}}\tau}S, \tag{6.19}$$

which has the solution in the series of harmonic functions,

$$S = S_0\cos\frac{x}{\sqrt{D_{\mathrm{a}}\tau}} \quad \text{and} \quad S = S_0\sin\frac{x}{\sqrt{D_{\mathrm{a}}\tau}}. \tag{6.20}$$

This solution must be supplied with the boundary conditions. Let's assume a simple model of plasma confined between two infinite plane-parallel walls (see Fig. 6.3), where the density of the plasma vanishes outside interval $x \in (-L, L)$. Then the solution to the problem is

$$n = n_0\left\{\sum_{l=0}^{\infty} a_{\mathrm{e},l}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},l}}}\cos\left[\frac{(l+\frac{1}{2})\pi x}{L}\right] + \sum_{m=1}^{\infty} a_{\mathrm{o},m}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},m}}}\sin\left[\frac{m\pi x}{L}\right]\right\} \tag{6.21}$$

The coefficients $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ may be determined by inserting the solution above into (6.16) and comparing the terms one by one. For the first term ($l = 0$) in the cosine series we have

$$-a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\frac{1}{\tau_{\mathrm{e},0}}\cos\frac{\pi x}{2L} = -D_{\mathrm{a}}a_{\mathrm{e},0}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{e},0}}}\left(\frac{\pi}{2L}\right)^2\cos\frac{\pi x}{2L} \tag{6.22}$$

and hence

$$\tau_{\mathrm{e},0} = \frac{4L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.23}$$

For a general mode l, we would similarly find

$$\tau_{\mathrm{e},l} = \frac{L^2}{(l+\frac{1}{2})^2\pi^2 D_{\mathrm{a}}}. \tag{6.24}$$

For the first term ($m = 1$) in the sine series, we have

$$-a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\frac{1}{\tau_{\mathrm{o},1}}\sin\frac{\pi x}{L} = -D_{\mathrm{a}}a_{\mathrm{o},1}\mathrm{e}^{-\frac{t}{\tau_{\mathrm{o},1}}}\left(\frac{\pi}{L}\right)^2\sin\frac{\pi x}{L} \tag{6.25}$$

and hence

$$\tau_{\mathrm{o},1} = \frac{L^2}{\pi^2 D_{\mathrm{a}}}. \tag{6.26}$$

For the other modes m we similarly obtain

$$\tau_{\mathrm{o},m} = \frac{L^2}{m^2\pi^2 D_{\mathrm{a}}}. \tag{6.27}$$

The lifetime of various modes $\tau_{\mathrm{e},l}$ and $\tau_{\mathrm{o},m}$ shows that the higher modes are attenuated faster. Therefore, if there is a wild density profile at the beginning (in time $t = 0$), then the profile is simplified with time. A set of snapshots of the evolution of an arbitrary density profile is given in Fig. 6.3.

Other physics-motivated examples for diffusion

The question is whether it is possible to achieve stationarity in the presence of diffusion. It is not if the continuity equation holds. It is possible in the presence of the source term,

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = Q. \tag{6.28}$$

In case of collisional ionisation $Q = Zn$, where Z is the ionisation function, we have

$$\boldsymbol{\nabla}\cdot n\boldsymbol{u} = -D_{\mathrm{a}}\,\triangle n = Zn \quad\to\quad \triangle n = -\frac{Z}{D_{\mathrm{a}}}n, \tag{6.29}$$

solution to which is again the series of harmonic functions.

The recombination, on the other hand, takes a different form because it does change the number of charged particles. The number density of the ionised particles naturally decays with n^2 in recombination, hence the continuity equation may be written in the form

$$\frac{\partial n}{\partial t} - D_{\mathrm{a}}\nabla n = -\alpha n^2. \tag{6.30}$$

Similarly to the previous cases, we consider the solution separated to the temporal and spatial parts $n = TS$. Further, we will only be interested in the temporal part, which is equivalent to modification of (6.30) to

$$\frac{\partial n}{\partial t} = -\alpha n^2, \tag{6.31}$$

where n^2 dependence is a consequence of the recombination being proportional to both the density of electrons and ions and the use of the plasma approximation. Then

$$\frac{1}{n^2}\mathrm{d}n = -\alpha \mathrm{d}t \quad \rightarrow \quad -\frac{1}{n} = -\frac{1}{n_0} - \alpha t. \tag{6.32}$$

For $t \to \infty$ we have $n \sim \frac{1}{\alpha t}$, which says that due to the recombination in plasmas, the density of the charged particles drops as $1/t$. The spatial part of the continuity equation is not relevant to the problem; thus, we neglect it from the solution.

This solution is important for understanding which process drives the diffusion of charged particles in a cloud of plasma left-alone in time. Obviously, in the initial moment, recombination will be the dominant process, as its time dependence is steeper for small t. On the contrary, when the density of the plasma gets smaller, the diffusion with exponencial time dependence prevails and further drives the diffusion.

6.2 Diffusion in highly ionised plasmas

We will show that in highly ionised plasmas, diffusion resembles the process of recombination rather than ambipolar diffusion. Let us use the fluid approximation to deal with the problem.

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p \tag{6.33}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{6.34}$$

Note that generally, the specific resistivity of plasma η is not a scalar value but rather a tensor and thus may depend on direction. In plasma, this usually is the case. Therefore, we split the specific resistivity into the parallel and perpendicular components ($\eta = \eta_{\|} + \eta_{\perp}$), where in the parallel direction Ohm's law simply is

$$\boldsymbol{E}_{\|} = \eta_{\|} \boldsymbol{j}_{\|}. \tag{6.35}$$

To obtain a similar relation in the direction perpendicular to the magnetic field, when assuming the stationarity, by vector-multiplying (6.34) by $\boldsymbol{B}$ and using (6.33) we have

$$\boldsymbol{E} \times \boldsymbol{B} + (\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} = \boldsymbol{E} \times \boldsymbol{B} - B^2 \boldsymbol{u} + B^2 \boldsymbol{u}_{\|} = \eta_{\perp} \nabla p. \tag{6.36}$$

Since $\boldsymbol{u} = \boldsymbol{u}_{\|} + \boldsymbol{u}_{\perp}$, then the last two terms on the left-hand side together give $B^2 \boldsymbol{u}_{\perp}$ and thus

$$\boldsymbol{u}_{\perp} = \frac{\boldsymbol{E} \times \boldsymbol{B}}{B^2} - \frac{\eta_{\perp} \nabla p}{B^2}. \tag{6.37}$$

The first term on the right-hand side is the same for both (electron and ion) fluids. The second one, on the other hand, may drive the diffusion. The perpendicular particle flux on top of the E-B drift is

$$\boldsymbol{\Gamma}_\perp = n\boldsymbol{u}'_\perp = -\frac{\eta_\perp n(K_\mathrm{B}T_\mathrm{e} + K_\mathrm{B}T_\mathrm{i})}{B^2}\nabla n = -D_\perp \nabla n\,. \tag{6.38}$$

Note that the diffusion coefficient $D_\perp$ is a function of particle density and for simpler manipulation may be rewritten as

$$D_\perp = \frac{\eta_\perp n K_\mathrm{B} \sum\limits_\alpha T_\alpha}{B^2} = 2An. \tag{6.39}$$

The factor of two is introduced so that $\boldsymbol{\nabla} \cdot (2An\nabla n) = 2A\boldsymbol{\nabla} \cdot (n\nabla n) = A \triangle n^2$.

The continuity equation

$$\frac{\partial n}{\partial t} + \boldsymbol{\nabla} \cdot (n\boldsymbol{u}) = 0 \tag{6.40}$$

then transforms to

$$\frac{\partial n}{\partial t} = A \triangle n^2. \tag{6.41}$$

Such an equation may be solved by separation of variables, hence $n(\boldsymbol{r}, t) = T(t)S(\boldsymbol{r})$. Then $S\frac{\mathrm{d}T}{\mathrm{d}t} = AT^2 \triangle S^2$ and

$$\frac{1}{T^2}\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{A}{S}\triangle S^2 = \mathrm{const} = -\frac{1}{\tau^2} \tag{6.42}$$

and in the case we solved previously. Then the temporal part has the solution

$$\frac{1}{T} = \frac{t}{\tau^2} + \frac{1}{T_0} \tag{6.43}$$

and the form of τ depends on the spatial part S. For $t \to \infty$

$$T \propto \frac{\tau^2}{t}, \tag{6.44}$$

which reminds us of the the solution of the change of the particle density caused by recombinations. The difference between the ambipolar diffusion and the result we just obtained is in the diffusion coefficient depending on particle density. Note that the result we obtained is not consistent with the assumptions of the derivation of Fick's law, as we allow for non-stationarity here.

6.3 Diffusion in a magnetic field

In presence of a magnetic field, the Euler equation retains an additional term. Let us see what happens in the presence of a magnetic field. We must go back to the double-fluid approximation since collisions are essential in the process of diffusion.

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right] = \operatorname{sgn} q\, en(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}) - K_\mathrm{B}T\nabla n - mn\nu\boldsymbol{u} \tag{6.45}$$

Let's assume that the collisions are much faster than the overall evolution of the system in time, or in other words $\frac{\mathrm{d}}{\mathrm{d}t} \ll \omega_\mathrm{c}, \nu$, where ν is the collision frequency with neutrals and $\tau \equiv 1/\nu$ is the corresponding time between collisions. Then the left-hand side may be neglected. Thus in components of the Cartesian system with the z axis oriented along the magnetic field ($\boldsymbol{B} = B\boldsymbol{e}_z$) we have

$$mn\nu u_x = \operatorname{sgn} q\, enE_x - K_\mathrm{B}T\frac{\partial n}{\partial x} + \operatorname{sgn} q\, enu_y B, \tag{6.46}$$

$$mn\nu u_y = \operatorname{sgn} q\, enE_y - K_\mathrm{B}T\frac{\partial n}{\partial y} - \operatorname{sgn} q\, enu_x B. \tag{6.47}$$

Along the magnetic field, a classic diffusion as derived above occurs.

By introducing $\mu = \frac{e}{m\nu}$, $D = \frac{K_\mathrm{B}T}{m\nu} = \frac{\omega_\mathrm{c}}{\nu}\frac{K_\mathrm{B}T}{eB}$ we have

$$u_x = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu} u_y, \tag{6.48}$$

$$u_y = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \operatorname{sgn} q\, \frac{\omega_\mathrm{c}}{\nu} u_x, \tag{6.49}$$

and (using $\tau \equiv 1/\nu$)

$$u_x(1 + \omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_\mathrm{c}^2\tau^2\frac{E_y}{B} - \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial y}, \tag{6.50}$$

$$u_y(1 + \omega_\mathrm{c}^2\tau^2) = \operatorname{sgn} q\, \mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} - \omega_\mathrm{c}^2\tau^2\frac{E_x}{B} + \operatorname{sgn} q\, \omega_\mathrm{c}^2\tau^2\frac{K_\mathrm{B}T}{neB}\frac{\partial n}{\partial x}. \tag{6.51}$$

The last two terms on the right-hand side obviously indicate the already known E-B and diamagnetic drifts. Hence

$$\boldsymbol{u}_\perp = \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)} + \operatorname{sgn} q \frac{\mu}{1 + \omega_\mathrm{c}^2\tau^2}\boldsymbol{E} - \frac{D}{1 + \omega_\mathrm{c}^2\tau^2}\frac{\nabla n}{n}, \tag{6.52}$$

where $\boldsymbol{v}_E = \boldsymbol{E} \times \boldsymbol{B}/B^2$ and $\boldsymbol{v}_\mathrm{D} = -\nabla p \times \boldsymbol{B}/(qnB^2)$. Let us further identify

$$\mu_\perp \equiv \frac{\mu}{1 + \omega_\mathrm{c}^2\tau^2} \quad \text{and} \quad D_\perp \equiv \frac{D}{1 + \omega_\mathrm{c}^2\tau^2}. \tag{6.53}$$

Hence generalised Fick's law transforms into

$$\boldsymbol{u}_\perp = \operatorname{sgn} q\, \mu_\perp \boldsymbol{E} - D_\perp\frac{\nabla n}{n} + \frac{\boldsymbol{v}_E + \boldsymbol{v}_\mathrm{D}}{1 + 1/(\omega_\mathrm{c}^2\tau^2)}. \tag{6.54}$$

The last term represents the already familiar drifts in the plasma fluid, only with the correction term for the diffusion. Obviously, the correcting term is important for $\nu > \omega_\mathrm{c}$. Hence in the weak field regime, drifts are suppressed by collisions.

The correction term for the mobility and the diffusion coeficient act in opposite ways. In the strong field, i.e., when $\nu < \omega_\mathrm{c}$, diffusion is suppressed. In particular, we have

$$D_\perp \propto \frac{K_\mathrm{B}T\nu}{m\omega_\mathrm{c}^2} \quad \text{and} \quad D_\parallel = \frac{K_\mathrm{B}T}{m\nu}, \tag{6.55}$$

where $D_\parallel$ is the non-modified diffusion coefficient which appears in the component of the Euler equation parallel to the magnetic field. Hence the decay caused by diffusion in plasmas has different rates in different directions. In case the collisional frequency is proportional[1] to $m^{-1/2}$ then the diffusion depends also on the species of the particles. Then

$$D_\perp \propto m^{1/2} \quad \text{and} \quad D_\parallel \propto m^{-1/2}. \tag{6.56}$$

In the perpendicular direction, the diffusion is faster for ions than electrons, while in the parallel direction the electrons diffuse faster. It has to be noted that generally speaking, collisions are important for particles to be able to penetrate across the magnetic field.

[1] As we discussed at the beginning of this chapter, $\nu = n_\mathrm{n}\langle\sigma v\rangle$. Leaving out a possible dependence of the effective cross-section on velocity, we have $\nu \propto \langle v\rangle \propto \sqrt{\langle v^2\rangle}$. Since $T \propto m\langle v^2\rangle$, for a fixed temperature T we have $\langle v^2\rangle \propto m^{-1}$ and finally $\nu \propto m^{-1/2}$.

Chapter 7

Equilibrium and stability

7.1 Equilibrium

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = \boldsymbol{j} \times \boldsymbol{B} - \nabla p. \tag{7.1}$$

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B} \tag{7.2}$$

$$\boldsymbol{B} \times \nabla p = \boldsymbol{B} \times (\boldsymbol{j} \times \boldsymbol{B}) = B^2 \boldsymbol{j} - \boldsymbol{B} \cdot \boldsymbol{j} \boldsymbol{B} = B^2 (\boldsymbol{j} - \boldsymbol{j}_{\parallel}). \tag{7.3}$$

$$\boldsymbol{j}_{\perp} = \frac{\boldsymbol{B} \times \nabla p}{B^2} = (K_{\mathrm{B}} T_{\mathrm{i}} + K_{\mathrm{B}} T_{\mathrm{e}}) \frac{\boldsymbol{B} \times \nabla n}{B^2}, \tag{7.4}$$

7.2 Plasma β

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.5}$$

$$\begin{aligned} \nabla p = \boldsymbol{j} \times \boldsymbol{B} = \frac{1}{\mu_0} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = \\ = \frac{1}{\mu_0} \left[-(\nabla \boldsymbol{B}) \cdot \boldsymbol{B} + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} \right] = \frac{1}{\mu_0} \left[(\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} - \frac{1}{2} \nabla B^2 \right] \end{aligned} \tag{7.6}$$

$$\nabla \left(p + \frac{1}{\mu_0} B^2 / 2 \right) = \frac{1}{\mu_0} (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B}. \tag{7.7}$$

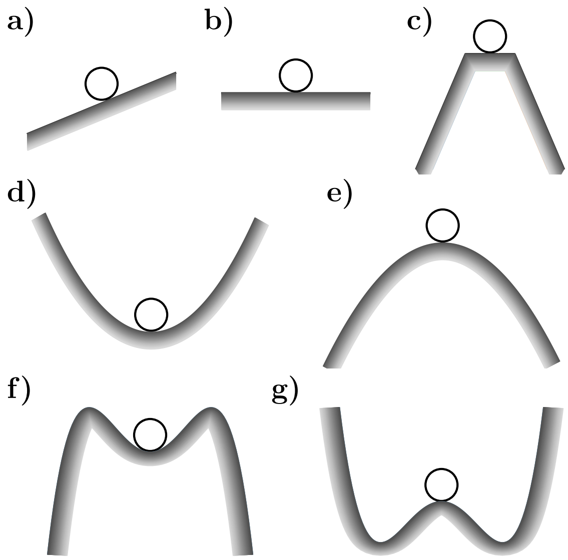
$$p + \frac{B^2}{2\mu_0} = \text{const.} \tag{7.8}$$

$$\beta \equiv \frac{p}{p_{\text{mag}}} = \frac{n \sum\limits_{\alpha} K_{\mathrm{B}} T_{\alpha}}{\frac{B^2}{2\mu_0}}, \tag{7.9}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \tag{7.10}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{j} \tag{7.11}$$

$$\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B} = \eta \boldsymbol{j} \tag{7.12}$$


a)
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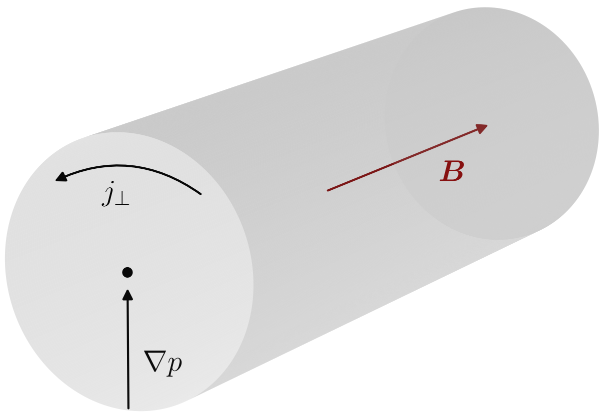
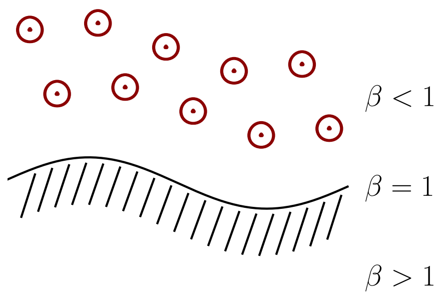

$j_\perp$
$\boldsymbol{B}$
∇p


$\beta < 1$
$\beta = 1$
$\beta > 1$

$$\boldsymbol{u} = \frac{1}{\rho}(n_\mathrm{i} M \boldsymbol{u}_\mathrm{i} + n_\mathrm{e} m \boldsymbol{u}_\mathrm{e}) \sim \frac{M\boldsymbol{u}_\mathrm{i} + m\boldsymbol{u}_\mathrm{e}}{M+m}, \tag{7.13}$$

$$\boldsymbol{j} = e(n_\mathrm{i}\boldsymbol{u}_\mathrm{i} - n_\mathrm{e}\boldsymbol{u}_\mathrm{e}) \sim ne(\boldsymbol{u}_\mathrm{i} - \boldsymbol{u}_\mathrm{e}). \tag{7.14}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\boldsymbol{\nabla} \times \eta \boldsymbol{j} = -\frac{\eta}{\mu_0}\left[\nabla(\boldsymbol{\nabla}\cdot\boldsymbol{B}) - \triangle \boldsymbol{B}\right] = \frac{\eta}{\mu_0}\triangle\boldsymbol{B}, \tag{7.15}$$

$$\frac{\partial \boldsymbol{B}}{\partial t} = \frac{\eta}{\mu_0 L^2}\boldsymbol{B}, \tag{7.16}$$

$$\boldsymbol{B} = \boldsymbol{B}_0 \exp\left[\pm\frac{t}{\tau}\right], \tag{7.17}$$

$$\tau = \frac{\mu_0 L^2}{\eta} \tag{7.18}$$

$$P = \eta j^2. \tag{7.19}$$

$$\Delta W = \eta j^2 \tau. \tag{7.20}$$

$$\Delta W = \eta j^2 \tau = \eta \left(\frac{B}{\mu_0 L}\right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0}. \tag{7.21}$$

7.3 Instabilities

$$Mn_0 \frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} = en_0 \boldsymbol{E}_1, \tag{7.22}$$

$$mn_0\left[\frac{\partial \boldsymbol{u}_{\mathrm{e}1}}{\partial t} + (\boldsymbol{u}_0\cdot\boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{e}1}\right] = -en_0\boldsymbol{E}_1. \tag{7.23}$$

$$-\mathrm{i}\omega M u_{\mathrm{i}1} = eE_1, \tag{7.24}$$

$$m(-\mathrm{i}\omega + \mathrm{i}ku_0)u_{\mathrm{e}1} = -eE_1. \tag{7.25}$$

$$\frac{\partial n_{\mathrm{i}1}}{\partial t} + n_0 \boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{i}1} = 0, \tag{7.26}$$

$$n_{\mathrm{i}1} = \frac{k}{\omega} n_0 u_{\mathrm{i}1} = \frac{\mathrm{i}en_0 k}{M\omega^2}E_1. \tag{7.27}$$

$$\frac{\partial n_{\mathrm{e}1}}{\partial t} + n_0\boldsymbol{\nabla}\cdot\boldsymbol{u}_{\mathrm{e}1} + (\boldsymbol{u}_0\cdot\boldsymbol{\nabla})n_{\mathrm{e}1} = 0, \tag{7.28}$$

$$(-\mathrm{i}\omega + \mathrm{i}ku_0)n_{\mathrm{e}1} + \mathrm{i}kn_0u_{\mathrm{e}1} = 0, \tag{7.29}$$

$$n_{\mathrm{e}1} = \frac{kn_0u_{\mathrm{e}1}}{\omega - ku_0} = -\frac{\mathrm{i}ekn_0}{m(\omega-ku_0)^2}E_1. \tag{7.30}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{E}_1 = \frac{e}{\varepsilon_0}(n_{\mathrm{i}1} - n_{\mathrm{e}1}), \tag{7.31}$$

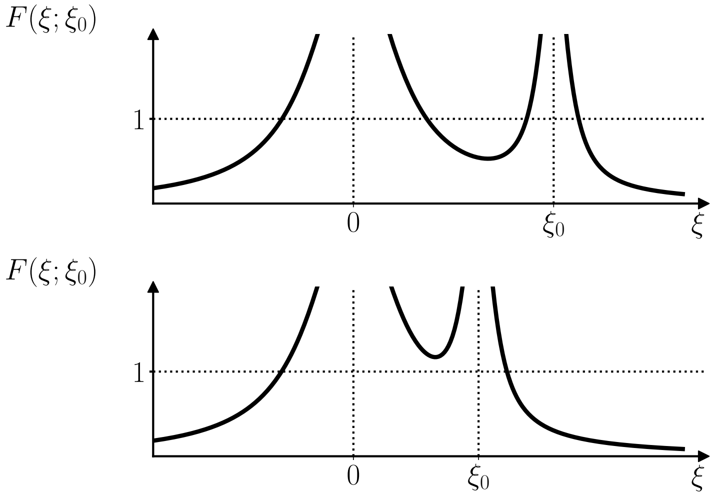
$$\mathrm{i}kE_1 = \frac{e}{\varepsilon_0}\mathrm{i}ekn_0\left[\frac{1}{M\omega^2} + \frac{1}{m(\omega-ku_0)^2}\right]E_1 \tag{7.32}$$

$$1 = \frac{e^2 n_0}{\varepsilon_0 m}\left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - k u_0)^2}\right] = \omega_\mathrm{p}^2\left[\frac{m/M}{\omega^2} + \frac{1}{(\omega - k u_0)^2}\right]. \tag{7.33}$$

$$\omega_j = \alpha_j + \mathrm{i}\gamma_j, \quad \text{where} \quad \alpha_j = \Re[\omega_j] \quad \text{and} \quad \gamma_j = \Im[\omega_j]. \tag{7.34}$$

$$E_1 = E\boldsymbol{e}_x \exp\left[\mathrm{i}(kx - \omega t)\right] \mathrm{e}^{\gamma_j t}. \tag{7.35}$$

$$1 = \frac{m/M}{\xi^2} + \frac{1}{(\xi - \xi_0)^2} = F(\xi; \xi_0). \tag{7.36}$$



$$M n_{\mathrm{i}0}[(\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}] = e n_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0} \times \boldsymbol{B}_0 + M n_{\mathrm{i}0}\boldsymbol{g}. \tag{7.37}$$

$$0 = e n_{\mathrm{i}0}\boldsymbol{u}_{\mathrm{i}0} \times \boldsymbol{B}_0 + M n_{\mathrm{i}0}\boldsymbol{g} \tag{7.38}$$

$$0 = e\boldsymbol{B}_0 \times (\boldsymbol{u}_{\mathrm{i}0} \times \boldsymbol{B}_0) + M\boldsymbol{B}_0 \times \boldsymbol{g} = eB_0^2\boldsymbol{u}_{\mathrm{i}0} - e\boldsymbol{B}_0 \cdot \boldsymbol{u}_{\mathrm{i}0}\boldsymbol{B}_0 + M\boldsymbol{B}_0 \times \boldsymbol{g}, \tag{7.39}$$

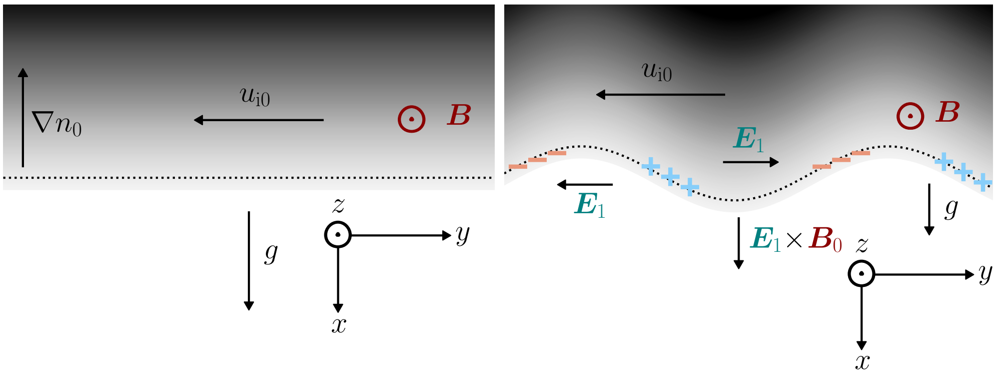
$$\boldsymbol{u}_{\mathrm{i}0} = -\frac{M}{e}\frac{\boldsymbol{B}_0 \times \boldsymbol{g}}{B_0^2} = -\frac{g}{\Omega_\mathrm{c}}\boldsymbol{e}_y, \tag{7.40}$$

$$M(n_{\mathrm{i}0} + n_{\mathrm{i}1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1}) + [(\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1}) \cdot \boldsymbol{\nabla}](\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1})\right\} = \\ e(n_{\mathrm{i}0} + n_{\mathrm{i}1})\left[\boldsymbol{E}_1 + (\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1}) \times \boldsymbol{B}_0\right] + M(n_{\mathrm{i}0} + n_{\mathrm{i}1})\boldsymbol{g} \tag{7.41}$$

$$M(n_{\mathrm{i}0} + n_{\mathrm{i}1})\left\{\frac{\partial}{\partial t}(\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1}) + (\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0} + (\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}1} + (\boldsymbol{u}_{\mathrm{i}1} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0}\right\} = \\ e(n_{\mathrm{i}0} + n_{\mathrm{i}1})\left[\boldsymbol{E}_1 + (\boldsymbol{u}_{\mathrm{i}0} + \boldsymbol{u}_{\mathrm{i}1}) \times \boldsymbol{B}_0\right] + M(n_{\mathrm{i}0} + n_{\mathrm{i}1})\boldsymbol{g} \tag{7.42}$$

$$M(n_{\mathrm{i}0} + n_{\mathrm{i}1})(\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}0} = (n_{\mathrm{i}0} + n_{\mathrm{i}1})e\boldsymbol{u}_{\mathrm{i}0} \times \boldsymbol{B}_0 + M(n_{\mathrm{i}0} + n_{\mathrm{i}1})\boldsymbol{g}. \tag{7.43}$$

$$M n_{\mathrm{i}0}\left[\frac{\partial \boldsymbol{u}_{\mathrm{i}1}}{\partial t} + (\boldsymbol{u}_{\mathrm{i}0} \cdot \boldsymbol{\nabla})\boldsymbol{u}_{\mathrm{i}1}\right] = e n_{\mathrm{i}0}(\boldsymbol{E}_1 + \boldsymbol{u}_{\mathrm{i}1} \times \boldsymbol{B}_0). \tag{7.44}$$



$$M(\omega - \boldsymbol{k}\cdot\boldsymbol{u}_{i0})\boldsymbol{u}_{i1} = \mathrm{i}e(\boldsymbol{E}_1 + \boldsymbol{u}_{i1}\times\boldsymbol{B}_0). \tag{7.45}$$

$$u_{i1x} = \frac{\mathrm{i}e}{M(\omega - ku_{i0})}u_{i1y}B_0, \tag{7.46}$$

$$u_{i1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{i0})}(E_{1y} - u_{i1x}B_0). \tag{7.47}$$

$$u_{i1y} = \frac{-\mathrm{i}e}{M(\omega - ku_{i0})}\left(E_{1y} + \frac{-\mathrm{i}eB_0^2}{M(\omega - ku_{i0})}u_{i1y}\right) \tag{7.48}$$

$$u_{i1y} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{i0})}\left[1 - \frac{e^2B_0^2/M^2}{(\omega - ku_{i0})^2}\right]^{-1} = \frac{-\mathrm{i}eE_{1y}}{M(\omega - ku_{i0})}\left[1 - \frac{\Omega_c^2}{(\omega - ku_{i0})^2}\right]^{-1}. \tag{7.49}$$

$$u_{i1x} = -\frac{e\Omega_c E_{1y}}{M(\omega - ku_{i0})^2}\left[1 - \frac{\Omega_c^2}{(\omega - ku_{i0})^2}\right]^{-1}. \tag{7.50}$$

$$u_{i1x} = \frac{E_{1y}}{B_0}, \quad u_{i1y} = -\mathrm{i}\frac{\omega - ku_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} \tag{7.51}$$

$$u_{e1x} = \frac{E_{1y}}{B_0}, \quad u_{e1y} = -\mathrm{i}\frac{\omega - ku_{e0}}{\omega_c}\frac{E_{1y}}{B_0}, \tag{7.52}$$

$$\frac{\partial n_{i1}}{\partial t} + \boldsymbol{\nabla}\cdot(n_{i0}\boldsymbol{u}_{i0}) + (\boldsymbol{u}_{i0}\cdot\boldsymbol{\nabla})n_{i1} + n_{i1}\boldsymbol{\nabla}\cdot\boldsymbol{u}_{i0} + (\boldsymbol{u}_{i1}\cdot\boldsymbol{\nabla})n_{i0} + n_{i0}\boldsymbol{\nabla}\cdot\boldsymbol{u}_{i1} + \boldsymbol{\nabla}\cdot(n_{i1}\boldsymbol{u}_{i1}) = 0. \tag{7.53}$$

$$-\mathrm{i}\omega n_{i1} + \mathrm{i}ku_{i0}n_{i1} + u_{i1x}\frac{\partial n_{i0}}{\partial x} + \mathrm{i}kn_{i0}u_{i1y} = 0. \tag{7.54}$$

$$-\mathrm{i}\omega n_1 + u_{e1x}\frac{\partial n_0}{\partial x} = 0. \tag{7.55}$$

$$-\mathrm{i}\omega n_1 + \mathrm{i}ku_{i0}n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} - \mathrm{i}^2kn_0\frac{\omega - ku_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0, \tag{7.56}$$

$$(\omega - ku_{i0})n_1 + \mathrm{i}\frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} + \mathrm{i}kn_0\frac{\omega - ku_{i0}}{\Omega_c}\frac{E_{1y}}{B_0} = 0. \tag{7.57}$$

$$-\mathrm{i}\omega n_1 + \frac{E_{1y}}{B_0}\frac{\partial n_0}{\partial x} = 0, \tag{7.58}$$

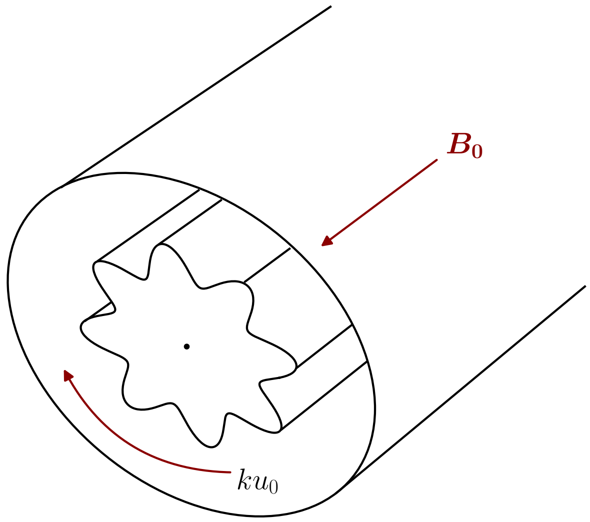


Figure 7.1: A flute instability.

$$(\omega - ku_{\mathrm{i0}})n_1 - \frac{\omega n_1}{\frac{\partial n_0}{\partial x}}\frac{\partial n_0}{\partial x} - kn_0\frac{\omega - ku_{\mathrm{i0}}}{\Omega_{\mathrm{c}}}\frac{\omega n_1}{\frac{\partial n_0}{\partial x}} = 0. \tag{7.59}$$

$$\omega(\omega - ku_{\mathrm{i0}}) = -\frac{u_{\mathrm{i0}}\Omega_{\mathrm{c}}}{n_0}\frac{\partial n_0}{\partial x}. \tag{7.60}$$

$$\omega(\omega - ku_{\mathrm{i0}}) = \frac{g}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.61}$$

$$\omega^2 - ku_{\mathrm{i0}}\omega - g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) = 0. \tag{7.62}$$

$$\omega = \frac{1}{2}ku_{\mathrm{i0}} \pm \left[\frac{1}{4}k^2u_{\mathrm{i0}}^2 + g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right)\right]^{\frac{1}{2}}. \tag{7.63}$$

$$-g\left(\frac{\frac{\partial n_0}{\partial x}}{n_0}\right) > \frac{1}{4}k^2u_{\mathrm{i0}}^2, \tag{7.64}$$

$$\boldsymbol{u}_{\mathrm{i0}} = \boldsymbol{u}_{\mathrm{Di}} = \frac{K_{\mathrm{B}}T_{\mathrm{i}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}\boldsymbol{e}_y, \tag{7.65}$$

$$\boldsymbol{u}_{\mathrm{e0}} = \boldsymbol{u}_{\mathrm{De}} = -\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}\boldsymbol{e}_y, . \tag{7.66}$$

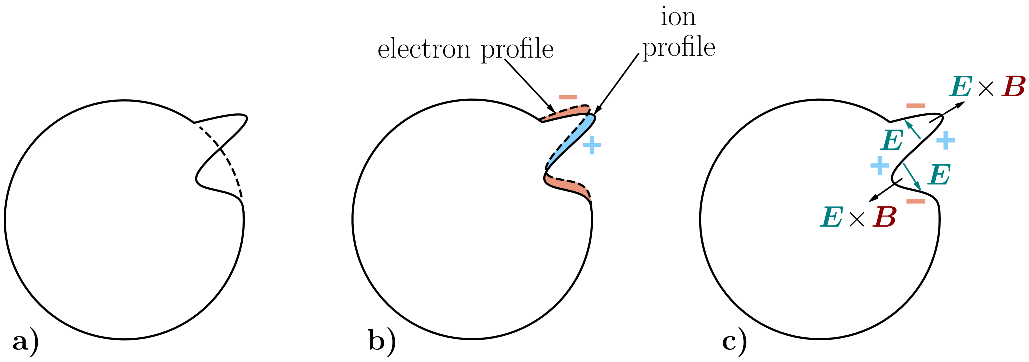
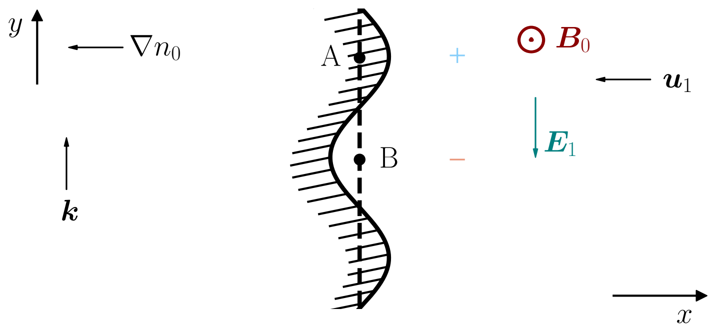


Figure 7.2: Development of the flute instability. a) Initial disturbance. b) Effect of ion and electron azimuthal drifts. c) Resulting $\boldsymbol{E}\times\boldsymbol{B}$ drifts increase the amplitude.



$$\frac{n_{\mathrm{e}1}}{n_0}=\frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}. \tag{7.67}$$

$$\boldsymbol{u}_1=\frac{\boldsymbol{E}_1\times\boldsymbol{B}_0}{B_0^2}. \tag{7.68}$$

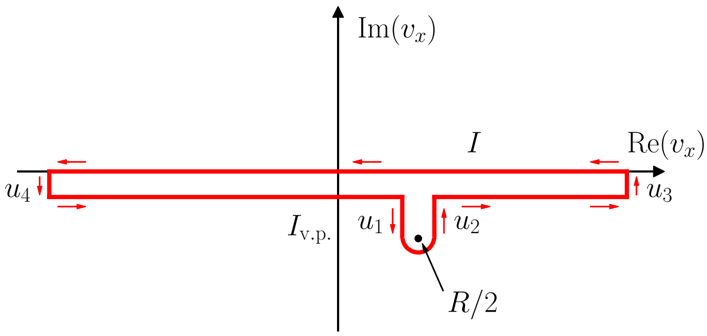
$$u_{1x}=\frac{E_y}{B_0}=-\frac{\mathrm{i}k_y\phi_1}{B_0}, \tag{7.69}$$

$$\frac{\partial n_1}{\partial t}=-u_{1x}\frac{\partial n_0}{\partial x}, \tag{7.70}$$

$$-\mathrm{i}\omega n_{\mathrm{e}1}=-u_{\mathrm{e}1x}\frac{\partial n_0}{\partial x}=\frac{\mathrm{i}k_y\phi_1}{B_0}\frac{\partial n_0}{\partial x}=-\mathrm{i}\omega n_0\frac{e\phi_1}{K_{\mathrm{B}}T_{\mathrm{e}}}, \tag{7.71}$$

$$\frac{\omega}{k}=-\frac{K_{\mathrm{B}}T_{\mathrm{e}}}{eB_0}\frac{1}{n_0}\frac{\partial n_0}{\partial x}, \tag{7.72}$$

$$\frac{\partial f}{\partial t}+\frac{\partial h}{\partial p_i}\frac{\partial f}{\partial r_i}-\frac{\partial h}{\partial r_i}\frac{\partial f}{\partial p_i}=\left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}}, \tag{7.73}$$



$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{F}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t} \right)_{\text{coll}}, \tag{7.74}$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{\langle F \rangle}{m} \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t} \right)'_{\text{coll}}, \tag{7.75}$$

$$\frac{\partial f}{\partial t} + \boldsymbol{v} \cdot \nabla f + \frac{q}{m} (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{7.76}$$

$$-\mathrm{i}\omega f_1 + \mathrm{i}k f_1 v_x = \frac{e}{m} E_{1x} \frac{\partial f_0}{\partial v_x}, \tag{7.77}$$

$$f_1 = \frac{\mathrm{i}eE_{1x}}{m} \frac{\partial f_0 / \partial v_x}{\omega - k v_x}. \tag{7.78}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{E}_1 = \mathrm{i}kE_{1x} = -\frac{e n_1}{\varepsilon_0} = -\frac{e}{\varepsilon_0} \int_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v}. \tag{7.79}$$

$$\int_{-\infty}^{+\infty} f_1 \mathrm{d}^3\boldsymbol{v} = \frac{\mathrm{i}eE_{1x}}{m} n_0 \int_{-\infty}^{+\infty} \frac{\partial \hat{f}_0 / \partial v_x}{\omega - k v_x} \mathrm{d}^3\boldsymbol{v} = -\frac{\mathrm{i}kE_{1x}\varepsilon_0}{e}, \tag{7.80}$$

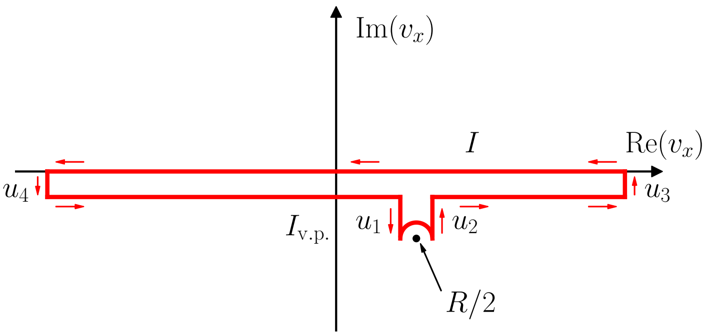
$$1 = -\frac{e^2 n_0}{\varepsilon_0 m k} \int_{-\infty}^{+\infty} \frac{\partial \hat{f}_0 / \partial v_x}{\omega - k v_x} \mathrm{d}^3\boldsymbol{v} = +\frac{\omega_{\mathrm{p}}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial \hat{f}_0 / \partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}^3\boldsymbol{v}. \tag{7.81}$$

$$\hat{f}_0(\boldsymbol{v}) = \tilde{f}_0(v_x)\tilde{f}_0(v_y)\tilde{f}_0(v_z), \tag{7.82}$$

$$1 = \frac{\omega_{\mathrm{p}}^2}{k^2} \int_{-\infty}^{+\infty} \mathrm{d}v_x \frac{\partial \tilde{f}_0(v_x) / \partial v_x}{v_x - \frac{\omega}{k}} \int_{-\infty}^{+\infty} \mathrm{d}v_y \tilde{f}_0(v_y) \int_{-\infty}^{+\infty} \mathrm{d}v_z \tilde{f}_0(v_z) = \frac{\omega_{\mathrm{p}}^2}{k^2} \int_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0 / \partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}v_x. \tag{7.83}$$

$$-I + I_{\text{v.p.}} + u_1 + u_2 + \frac{R}{2} + u_3 + u_4 = 0, \tag{7.84}$$

$$1 = \frac{\omega_{\mathrm{p}}^2}{k^2} \left\{ \text{v.p.} \int_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0 / \partial v_x}{v_x - \frac{\omega}{k}} \mathrm{d}v_x + \left[\mathrm{i}\pi \frac{\partial \tilde{f}_0}{\partial v_x} \right]_{\frac{\omega}{k}} \right\}. \tag{7.85}$$



$$\int_{-\infty}^{+\infty} \frac{\partial \tilde{f}_0}{\partial v_x} \frac{\mathrm{d}v_x}{v_x - \omega/k} = \left[\frac{\tilde{f}_0}{v_x - \omega/k}\right]_{+\infty}^{-\infty} + \int_{-\infty}^{+\infty} \frac{\tilde{f}_0 \mathrm{d}v_x}{(v_x - \omega/k)^2}. \tag{7.86}$$

$$(v_x - v_\varphi)^{-2} \sim v_\varphi^{-2}\left(1 + \frac{2v_x}{v_\varphi} + \frac{3v_x^2}{v_\varphi^2}\right). \tag{7.87}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\tilde{f}_0 \mathrm{d}v_x}{(v_x - v_\varphi)^2} \sim v_\varphi^{-2}\left[\int_{-\infty}^{+\infty} \tilde{f}_0 \mathrm{d}v_x + \int_{-\infty}^{+\infty} \frac{2v_x}{v_\varphi}\tilde{f}_0 \mathrm{d}v_x + \int_{-\infty}^{+\infty} \frac{3v_x^2}{v_\varphi^2}\tilde{f}_0 \mathrm{d}v_x\right] = \\ = v_\varphi^{-2}\left[1 + \frac{2}{v_\varphi}\langle v_x\rangle + \frac{3\langle v_x^2\rangle}{v_\varphi^2}\right] = v_\varphi^{-2}\left(1 + \frac{3\langle v_x^2\rangle}{v_\varphi^2}\right),\end{aligned} \tag{7.88}$$

$$1 = \frac{\omega_\mathrm{p}^2}{k^2}\frac{k^2}{\omega^2}\left\{1 + \frac{3\langle v_x^2\rangle}{v_\varphi^2} + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.89}$$

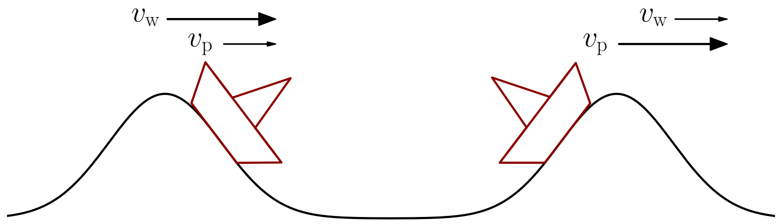
$$1 = \frac{\omega_\mathrm{p}^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}\frac{K_\mathrm{B}T_\mathrm{e}}{m}\right). \tag{7.90}$$

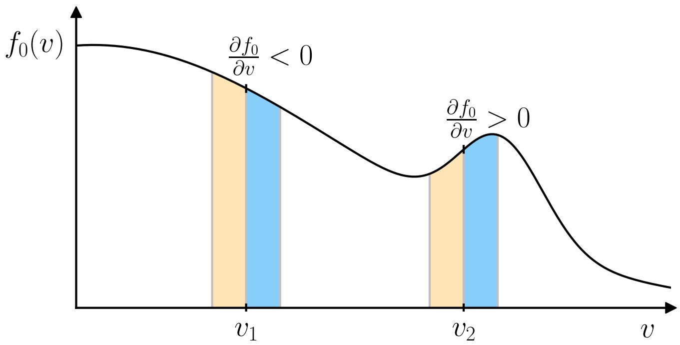
$$\omega^2 = \omega_\mathrm{p}^2 + 3k^2\frac{K_\mathrm{B}T_\mathrm{e}}{m}, \tag{7.91}$$

$$\omega^2 = \omega_\mathrm{p}^2\left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.92}$$

$$\omega \sim \omega_\mathrm{p}\left\{1 + \frac{1}{2}\mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\}. \tag{7.93}$$

$$\frac{\omega^2}{\omega_\mathrm{p}^2} = \left\{1 + \mathrm{i}\pi\left[\frac{\omega^2}{k^2}\frac{\partial \tilde{f}_0}{\partial v_x}\right]_{\frac{\omega}{k}}\right\} \sim 1 \tag{7.94}$$





$$\omega \sim \omega_\mathrm{p} \left\{ 1 + \frac{1}{2} \mathrm{i}\pi \left[\frac{\omega_\mathrm{p}^2}{k^2} \frac{\partial \tilde{f}_0}{\partial v_x} \right]_{\frac{\omega}{k}} \right\}. \tag{7.95}$$

$$\Im \left(\frac{\omega}{\omega_\mathrm{p}} \right) \sim 0.22\sqrt{\pi} \left(\frac{\omega_\mathrm{p}}{k v_\mathrm{T}} \right)^3 \exp\left[-\frac{1}{2k^2\lambda_\mathrm{D}} \right], \tag{7.96}$$