N-Body Methods and Algorithms

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Newton's Equation

Force $$\mathbf{F}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

Explicit differentiation

$$\dot{\mathbf{F}}_i = -G \sum_{j=1;\, j\neq i}^{N} m_j \frac{\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

$$- 3m_j \frac{(\mathbf{r}_i - \mathbf{r}_j)\cdot(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^2} \frac{\mathbf{r}_i - \mathbf{r}_j}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

New solution at $t = \Delta t$

$$\Delta\dot{\mathbf{r}}_i = (\frac{1}{2}\dot{\mathbf{F}}_i \Delta t + \mathbf{F}_i)\Delta t$$

$$\Delta\mathbf{r}_i = ((\frac{1}{6}\dot{\mathbf{F}}_i \Delta t + \frac{1}{2}\mathbf{F}_i)\Delta t + \dot{\mathbf{r}}_i)\Delta t$$

Repeat cycle for $i = 1, N$; N^2 interactions

Hermite Integration

Taylor series for $\mathbf{F}$ and $\mathbf{F}^{(1)}$

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\,t + \frac{1}{2}\mathbf{F}_0^{(2)}\,t^2 + \frac{1}{6}\mathbf{F}_0^{(3)}\,t^3$$

$$\mathbf{F}^{(1)} = \mathbf{F}_0^{(1)} + \mathbf{F}_0^{(2)}\,t + \frac{1}{2}\mathbf{F}_0^{(3)}\,t^2$$

Prediction

$$\mathbf{r}_j = ((\frac{1}{6}\mathbf{F}_0^{(1)}\,\delta t'_j + \frac{1}{2}\mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0)\,\delta t'_j + \mathbf{r}_0$$

$$\mathbf{v}_j = ((\frac{1}{2}\mathbf{F}_\mathbf{0}^{(1)}\,\delta t'_j + \mathbf{F}_0)\,\delta t'_j + \mathbf{v}_0\,; \qquad \delta t'_j = t - t_0$$

New forces $\quad \mathbf{F},\ \mathbf{F}^{(1)}$

Higher derivatives

$$\mathbf{F}_0^{(3)} = (2(\mathbf{F}_0 - \mathbf{F}) + (\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{6}{t^3}$$

$$\mathbf{F}_0^{(2)} = (-3(\mathbf{F}_0 - \mathbf{F}) - (2\mathbf{F}_0^{(1)} + \mathbf{F}^{(1)})\,t)\,\frac{2}{t^2}$$

Corrector for i

$$\Delta\mathbf{r}_i = \frac{1}{24}\mathbf{F}_0^{(2)}\Delta t^4 + \frac{1}{120}\mathbf{F}_0^{(3)}\,\Delta t^5$$

$$\Delta\mathbf{v}_i = \frac{1}{6}\mathbf{F}_0^{(2)}\Delta t^3 + \frac{1}{24}\mathbf{F}_0^{(3)}\,\Delta t^4$$

Basic Regularization

Two-body equation $\ddot{x} = -\frac{M}{x^2}$

Smoothing function $t' \equiv \frac{dt}{d\tau} = x$

Rule of differentiation $\frac{d}{dt} = \frac{1}{x}\frac{d}{d\tau}$

Time-smoothed equation $x'' = \frac{x'^2}{x} - M$

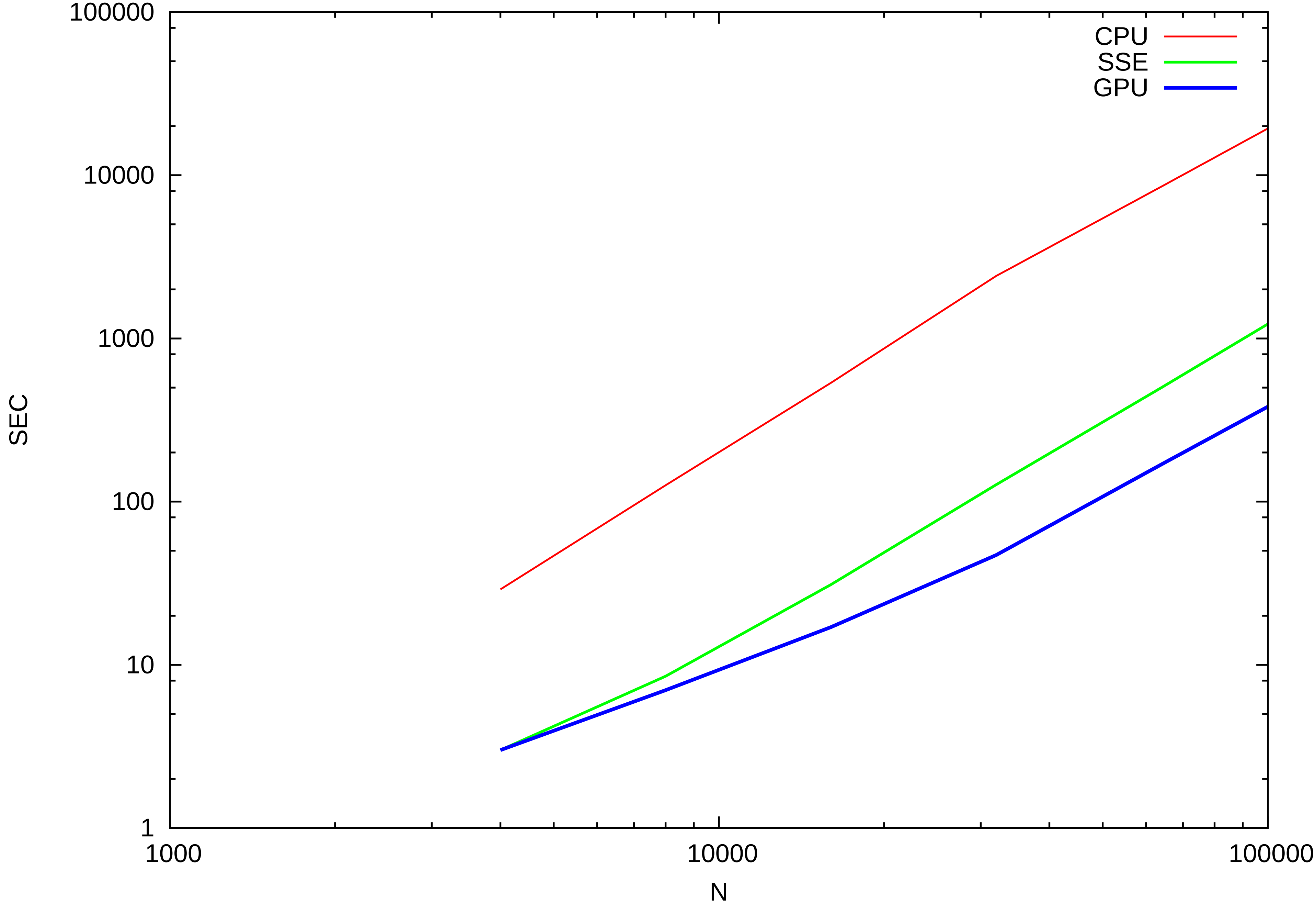
Binding energy $h = \frac{1}{2}\dot{x}^2 - \frac{M}{x}$

Substitution $\dot{x} = \frac{x'}{x} \Rightarrow$ $x'' = 2hx + M$

Coordinate transformation $u^2 = x$

Twice diff. of u^2 and $h \Rightarrow$ $u'' = \frac{1}{2}hu$

Regular equation for $x \Rightarrow 0$



CPU
SSE
GPU
100000
10000
1000
100
10
1
SEC
1000
10000
100000
N

Neighbour Scheme

Total force $\mathbf{F}(t) = \sum_{j=1}^{n} \mathbf{F}_j + \mathbf{F}_d(t)$

Prediction & $\dot{\mathbf{F}}$

$$\mathbf{F}(t) = \mathbf{F}_n + \dot{\mathbf{F}}_d(t - t_0) + \mathbf{F}_d(t_0), \quad \dot{\mathbf{F}} = \dot{\mathbf{F}}_n + \dot{\mathbf{F}}_d$$

Time-scales $\Delta t_n << \Delta t_d, \quad n << N$

Contrast $C = \frac{2n}{N}\left(\frac{r_h}{R_s}\right)^3, \quad n_p = n_{max}(0.04 * C)^{1/2}$

Neighbour sphere $R_s^{\mathrm{new}} = R_s^{\mathrm{old}}\left(\frac{n_p}{n}\right)^{1/3}$

Neighbour selection $|\mathbf{r}_i - \mathbf{r}_j| < R_s, \quad \Rightarrow \ list$

Derivative corrections $\mathbf{F}_{ij}^{(2)}, \mathbf{F}_{ij}^{(3)}$

Time-Steps

Basic time-step $\Delta t = \frac{\alpha|\mathbf{r}|}{|\mathbf{v}|}, \quad \Delta t = \frac{\beta|\mathbf{F}|}{|\mathbf{F}^{(1)}|}$

Taylor series $\mathbf{F} = \mathbf{F}_0 + \mathbf{F}_0^{(1)}\Delta t + \frac{1}{2}\mathbf{F}_0^{(2)}\Delta t^2 + ...$

Natural time-step $\Delta t = \left(\frac{\eta|\mathbf{F}|}{|\mathbf{F}^{(2)}|}\right)^{1/2}, \quad \eta = 0.02$

General expression $\Delta t = \left(\frac{\eta(|\mathbf{F}||\mathbf{F}^{(2)}| + |\mathbf{F}^{(1)}|^2)}{|\mathbf{F}^{(1)}||\mathbf{F}^{(3)}| + |\mathbf{F}^{(2)}|^2}\right)^{1/2}$

Relative criterion Δt independent of mass

Block-steps $\Delta t_n = \frac{\Delta t_1}{2^{n-1}}, \quad \Delta t_1 = 1$

Hierarchical levels $\mathcal{N}_k$ particles with steps Δt_k

Scheduling $i = \min\,(t_j + \Delta t_j)$

Levi-Civita Formulation

2D system: u_1, u_2

$$\begin{aligned} R_1 &= u_1^2 - u_2^2 \\ R_2 &= 2u_1 u_2 \end{aligned}$$

Transformation

$$\mathbf{R} = \mathcal{L}(\mathbf{u})\mathbf{u}$$

Levi-Civita [1920] matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{bmatrix} \qquad \Rightarrow \; R = u_1^2 + u_2^2$$

Definition $\dot{\mathbf{R}}^2 = \dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}}$ with $\mathbf{R}' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'$ and $\dot{R} = R'/R$

$$\dot{\mathbf{R}} = 2\mathcal{L}(\mathbf{u})\mathbf{u}'/R$$

$\dot{\mathbf{R}}^T = 2\mathbf{u}'\mathcal{L}^T(\mathbf{u})/R$ and $\mathcal{L}^T(\mathbf{u})\mathcal{L}(\mathbf{u}) = R\mathbf{I}$ give

$$\dot{\mathbf{R}}^T \cdot \dot{\mathbf{R}} = 4\mathbf{u}' \cdot \mathbf{u}'/R$$

From $\mathcal{L}'(\mathbf{u}) = \mathcal{L}(\mathbf{u}')$ we have $\mathbf{R}'' = 2\mathcal{L}(\mathbf{u})\mathbf{u}'' + 2\mathcal{L}(\mathbf{u}')\mathbf{u}'$

Final equation of motion, with $\mathbf{u} \cdot \mathbf{u} = R$

$$\mathbf{u}'' = \frac{1}{2}h\mathbf{u} + \frac{1}{2}R\mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

Binding energy per unit reduced mass

$$h = [(2\mathbf{u}' \cdot \mathbf{u}' - (m_k + m_l)]/R$$

Rate of change from $\dot{\mathbf{R}} \cdot \ddot{\mathbf{R}}$

$$\frac{d}{dt}\left[\frac{1}{2}\dot{\mathbf{R}}^2 - \frac{(m_k + m_l)}{R}\right] = \dot{\mathbf{R}} \cdot \mathbf{F}_{kl}$$

Conversion by $h' = \mathbf{R}' \cdot \mathbf{F}_{kl}$ and $\dot{\mathbf{R}}$

$$h' = 2\mathbf{u}' \cdot \mathcal{L}^T(\mathbf{u})\mathbf{F}_{kl}$$

KS Regularization

New coordinates $R = u_1^2 + u_2^2 + u_3^2 + u_4^2$

Time transformation $dt = R\,d\tau$

Coordinate transformation $\mathbf{R} = \mathcal{L}(\mathbf{u})\,\mathbf{u}$

Levi-Civita matrix

$$\mathcal{L}(\mathbf{u}) = \begin{bmatrix} u_1 & -u_2 & -u_3 & u_4 \\ u_2 & u_1 & -u_4 & -u_3 \\ u_3 & u_4 & u_1 & u_2 \end{bmatrix}$$

Equations of motion

$$\begin{aligned} \mathbf{u}'' &= \frac{1}{2}h\,\mathbf{u} + \frac{1}{2}R\,\mathcal{L}^{\mathcal{T}}\,\mathbf{P} \\ h' &= 2\,\mathbf{u}' \cdot \mathcal{L}^{\mathcal{T}}\,\mathbf{P} \\ t' &= \mathbf{u} \cdot \mathbf{u} \end{aligned}$$

Close encounter $\Delta t_i < \Delta t_{cl}\,; \quad R < r_{cl}$

Termination $\gamma \equiv \dfrac{|\mathbf{P}|\,R^2}{m_i + m_j} > 0.5$

Centre of mass motion $\ddot{\mathbf{r}} = \dfrac{m_i\,\mathbf{P}_i + m_j\,\mathbf{P}_j}{m_i + m_j}$

Perturber selection $r_k < \lambda\,R\,, \quad \gamma > 1 \times 10^{-6}$

KS Initialization

Case (i) $R_1 \leq 0$: add R_1 to R,

$$u_1^2 + u_4^2 = \tfrac{1}{2}(R_1 + R)\,.$$

Redundancy choice $u_4 = 0$ gives

$$\begin{aligned} u_1 &= \left[\tfrac{1}{2}(R_1 + R)\right]^{1/2}, \\ u_2 &= \tfrac{1}{2}R_2/u_1\,, \\ u_3 &= \tfrac{1}{2}R_3/u_1\,. \end{aligned}$$

Case (ii) $R_1 < 0$: subtract R_1 from R,

$$u_2^2 + u_3^2 = \tfrac{1}{2}(R - R_1)\,.$$

Setting $u_3 = 0$ leads to

$$\begin{aligned} u_2 &= \left[\tfrac{1}{2}(R - R_1)\right]^{1/2}, \\ u_1 &= \tfrac{1}{2}R_2/u_2\,, \\ u_4 &= \tfrac{1}{2}R_3/u_2\,. \end{aligned}$$

Derivative of $\mathbf{R}$ and property of $\mathcal{L}(\mathbf{u})$ yields

$$\mathbf{R}' = 2\mathcal{L}\mathbf{u})\mathbf{u}'\,.$$

Apply $\mathcal{L}^T(\mathbf{u})$ on both sides with $\mathcal{L}^T\mathcal{L} = R$,

$$\mathbf{u}' = \tfrac{1}{2}\mathcal{L}(\mathbf{u})\mathbf{R}'/R\,.$$

Definition $R' = R\dot{R}$ gives the KS velocity

$$\mathbf{u}' = \tfrac{1}{2}\mathcal{L}(\mathbf{u})\dot{\mathbf{R}}\,.$$

Hermite KS

Standard KS

$$\begin{aligned}
\mathbf{u}'' &= \frac{1}{2} h\,\mathbf{u} + \frac{1}{2} R\,\mathcal{L}^T\,\mathbf{F}_{kl} \\
h' &= 2\,\mathbf{u}' \cdot \mathcal{L}^T\,\mathbf{F}_{kl} \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}$$

New notation

$$\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \mathbf{u}'' \\
\mathbf{Q} &= \mathcal{L}^T\,\mathbf{P}\,,
\end{aligned}$$

with $\mathbf{P} = \mathbf{F}_{kl}$ as the perturbing force.

Basic equations

$$\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \frac{1}{2} h\,\mathbf{u} + \frac{1}{2} R\,\mathbf{Q} \\
h' &= 2\,\mathbf{u}' \cdot \mathbf{Q} \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}$$

Hermite $\mathbf{F}$, $\mathbf{F}'$ formulation

$$\begin{aligned}
\mathbf{F}_{\mathrm{u}} &= \frac{1}{2} h\,\mathbf{u} + \frac{1}{2} R\,\mathbf{Q} \\
\mathbf{F}'_{\mathrm{u}} &= \frac{1}{2}(h'\mathbf{u} + h\mathbf{u}' + R'\mathbf{Q} + R\mathbf{Q}') \\
h' &= 2\,\mathbf{u}' \cdot \mathbf{Q} \\
h'' &= 2\mathbf{F}_{\mathrm{u}} \cdot \mathbf{Q} + 2\mathbf{u}' \cdot \mathbf{Q}' \\
t' &= \mathbf{u} \cdot \mathbf{u}
\end{aligned}$$

The derivatives of $\mathbf{P}$, $\mathbf{Q}$ and t' are readily available. Note that $\mathbf{P}' = R\dot{\mathbf{P}}$ and that $\mathcal{L}^T(\mathbf{u}')$ can be obtained by substituting $\mathbf{u}'$ for $\mathbf{u}$. For implementation, significant accuracy can be gained by high-order prediction (not used in standard Hermite).

KS Decision-Making

Close encounter	$R_{\rm cl} = \dfrac{4\, r_{\rm h}}{N\, C^{1/3}}, \quad \Delta t_{\rm cl} = \beta \left(\dfrac{R_{\rm cl}^3}{\bar m}\right)^{1/2}$
Time-step criterion	$\Delta t_k < \Delta t_{\rm cl}$
Neighbour list search	list all $r_{kj}^2, \quad \Delta t_j < 2\,\Delta t_{\rm cl}$
Two-body selection	$R_{kl} < R_{\rm cl}, \quad \dot R_{kl} < 0$
Dominant motion	$\dfrac{m_k + m_l}{R_{kl}^2} > \dfrac{m_k + m_j}{R_{kj}^2}$
KS initialization	$\mathbf{F}_U,\ \mathbf{F}'_U,\ \Delta\tau\ \&\ t^{(n)} \ \Rightarrow\ \Delta t$
Initialization of c.m.	$\mathbf{r}_{\rm cm} = \dfrac{m_k\,\mathbf{r}_k + m_l\,\mathbf{r}_l}{m_k + m_l}$
Perturber search	$r_{\rm p} < \left(\dfrac{2m_{\rm p}}{m_{\rm b}\gamma_{\rm min}}\right)^{1/3} a\,(1+e)$
Slow-down adjustment	$\gamma < \gamma_0, \quad \Delta\tau \Rightarrow \kappa\,\Delta t$
Termination test	$R > R_0, \quad \gamma > \gamma^*$
Delayed termination	$T_{\rm block} - t > \Delta t_i$
Final iteration	$\Delta\tau$ from $\dot\tau, \ddot\tau, \ldots$ and δt
Polynomial initialization	$\mathbf{F}_j,\ \dot{\mathbf{F}}_j,\ \Delta t_j, \quad j = k, l$

Practical Aspects of KS

Regular equations	Perturbed harmonic oscillator, $\gamma < 1$
Constant time-step	$\Delta\tau = \eta \left(\frac{1}{2\lvert h \rvert}\right)^{1/2}$ vs $\Delta t \propto R^{3/2}$
Linearized equations	Higher accuracy per step
Faster force calculation	Tidal perturbation, $P \propto 1/r^3$
Unperturbed motion	$\gamma < 10^{-6}$, $\Delta t > t_{\mathrm{K}}$
Slow-down procedure	Adiabatic invariance, $\tilde{P} = \kappa P$
Energy rectification	Improve $\mathbf{u}, \mathbf{u}'$ from integration of h'
C.m. approximation	$d > 100\, a\, (1 + e)$
Transformations	$\mathbf{R} = \mathcal{L}\mathbf{u}$, $\mathbf{r}_j = \mathbf{r}_{\mathrm{cm}} \pm \mu\mathbf{R}/m_j$
	$\dot{\mathbf{R}} = 2\mathcal{L}\mathbf{u}'/R$, $\dot{\mathbf{r}}_j = \dot{\mathbf{r}}_{\mathrm{cm}} \pm \mu\dot{\mathbf{R}}/m_j$
Two-body elements	$a, \mathbf{J}, e$ for averaging & circularization

Scaling of Initial Conditions

Main input	$N, \;\; N_{\rm b}, \;\; M_{\rm S}, \;\; R_{\rm pc}$
Cluster parameters	optional IMF and Plummer or King model
Initial data	$\tilde{m}_i, \, \tilde{\mathbf{r}}_i, \, \tilde{\mathbf{v}}_i, \, ..., \; i = 1, \, N$
Total energy	$E \; = \; T \; - \; U$
Virial theorem	$\mathbf{v}_i = q \, \tilde{\mathbf{v}}_i, \;\; q \; = \; \left[\dfrac{Q_{\rm V} U}{T} \right]^{1/2}, \;\; \mathbf{r}_i \; = \; \tilde{\mathbf{r}}_i$
Standard units	$G = 1, \;\; \sum m_i \; = \; 1, \;\; E_0 \; = \; -0.25$
Standard scaling	$\hat{\mathbf{r}}_i \; = \; \dfrac{\mathbf{r}_i}{S^{1/2}}, \; \hat{\mathbf{v}}_i \; = \; \mathbf{v}_i \, S^{1/2}, \; S \; = \; \dfrac{E_0}{q^2 T - U}$
Astrophysical units	$V^*, \, T^*, \, R^*$ from $M_{\rm tot}, \, R_{\rm pc}$
Primordial binaries	split or copy m_i, introduce $a, \, e, \, \Omega$
Force polynomials	$\mathbf{F}_i, \, \dot{\mathbf{F}}_i, \, \Delta t_i, \, ..., \;\; i = 1, \, N$
KS regularization	explicit initialization, $R \; < \; R_{\rm cl}$

Units

(a) Scaling relations

Given length scale R_V in pc and total mass NM_S in $M_\odot$

Velocity scaling

$\tilde{V}^* = 1 \times 10^{-5} (GM_\odot / L^*)^{1/2}$ km/s, with $L^* = 3 \times 10^{18}$ cm

Velocity unit $\quad V^* = 6.557 \times 10^{-2} (NM_\mathrm{S}/R_\mathrm{V})^{1/2}$ km/s

Fiducial time $\quad \tilde{T}^* = (L^{*3}/GM_\odot)^{1/2} = 14.94$ Myr

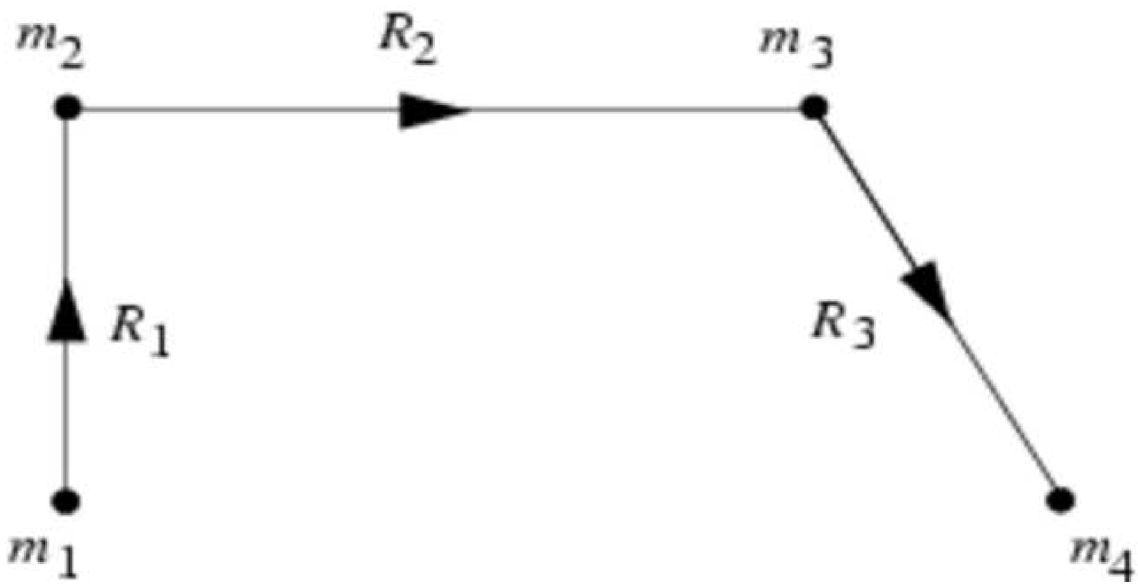
Time unit $\quad T^* = 14.94\,(R_\mathrm{V}^3/NM_\mathrm{S})^{1/2}$ Myr

(b) Conversion from N-body to physical units

$\tilde{r} = R_\mathrm{V}\, r$ pc, $\quad \tilde{v} = V^*\, v\ km/s, \quad \tilde{t} = T^*\, t$ Myr,

$\tilde{m} = NM_\mathrm{S}\, m\ M_\odot$

Crossing time $\quad T_\mathrm{cr} = 2\sqrt{2}\, T^*$ Myr


m_2
R_2
m_3
R_1
R_3
m_1
m_4

Data Structure

Single stars	$2\,N_{\rm p} < i \le N, \quad \mathcal{N}_i = i$
KS pairs	$1 \le i \le 2\,N_{\rm p}\,, \quad i_p = i_{\rm icm} - N$
C.m. particles	$i > N, \quad \mathcal{N} = N_0 + \mathcal{N}_k\,, \quad k = 2i_p - 1$
Stable triples	$\mathrm{KS} + \mathrm{ghost}, \quad \mathcal{N}_{\rm cm} = -\mathcal{N}_k$
Ghost particles	$\mathcal{N}_{\rm ghost} = \mathcal{N}_{2i_{\rm p}-1}\,, \quad m_{\rm ghost} = 0$
Stable quadruples	$\mathrm{KS} + \mathrm{KS\ ghost}, \quad \mathcal{N}_{\rm cm} = -\mathcal{N}_k$
Higher orders	$\mathrm{T} + \mathrm{KS}, \quad \mathcal{N}_{\rm cm} = -(2N_0 + \mathcal{N}_k)$
Chain members	$2\,N_{\rm p} < i_{\rm cm} \le N, \quad \mathcal{N}_{\rm cm} = 0$
Single escape	$2N_{\rm p} < i \le N, \quad r_i > 2r_{\rm tide}\,, \quad \mathrm{remove}\ i$
Binary escape	$i > N\,, \quad r_i > 2r_{\rm tide}\,, \quad 2i_p - 1,\ 2i_p$
Hierarchy escape	$i > N\,, \quad r_i > 2r_{\rm tide}\,, \quad 2i_p - 1,\ 2i_p\,,\ i_{\rm ghost}$

Energy Budget

Definition of total energy

$$E_{\rm tot} = T + U + E_{\rm tide} + E_{\rm bin} + E_{\rm merge} + E_{\rm coll} + E_{\rm mdot} + E_{\rm cdot} + E_{\rm ch} + E_{\rm sub}$$

T	Kinetic energy of single bodies and c.m. particles
U	Potential energy of single and c.m. bodies
$E_{\rm tide}$	Tidal energy due to external perturbations
$E_{\rm bin}$	Binding energy in regularized pairs
$E_{\rm merge}$	Total internal energy of hierarchical systems
$E_{\rm coll}$	Sum of binding energies released in collisions
$E_{\rm mdot}$	Energy change from mass loss and Roche mass transfer
$E_{\rm cdot}$	Neutron star kicks and common envelope evolution
$E_{\rm ch}$	Total energy of any existing chain subsystem
$E_{\rm sub}$	Energy of unperturbed triple and quadruple subsystems
ΔE	Energy change due to removal of escapers

NBODY6 Input File

1 20.0

1000 1 5 50000 95 1

0.01 0.01 0.3 2.0 10.0 100.0 1.0D-04 1.0 0.5

0 0 0 0 1 0 1 0 0 0

0 0 0 1 1 1 0 1 3 4

1 0 2 0 0 0 0 0 0 1

0 0 0 0 0 0 0 0 0 1

0 0 0 0 0 0 0 0 0 0

4.0D-05 4.0D-04 0.2 1.0 1.0D-06 0.001

2.3 10.0 0.2 0 0 0.02 0.0 100.0

0.5 0 0 0 0.125

KSTART TCOMP

N NFIX NCRIT NRAND NNBMAX NRUN

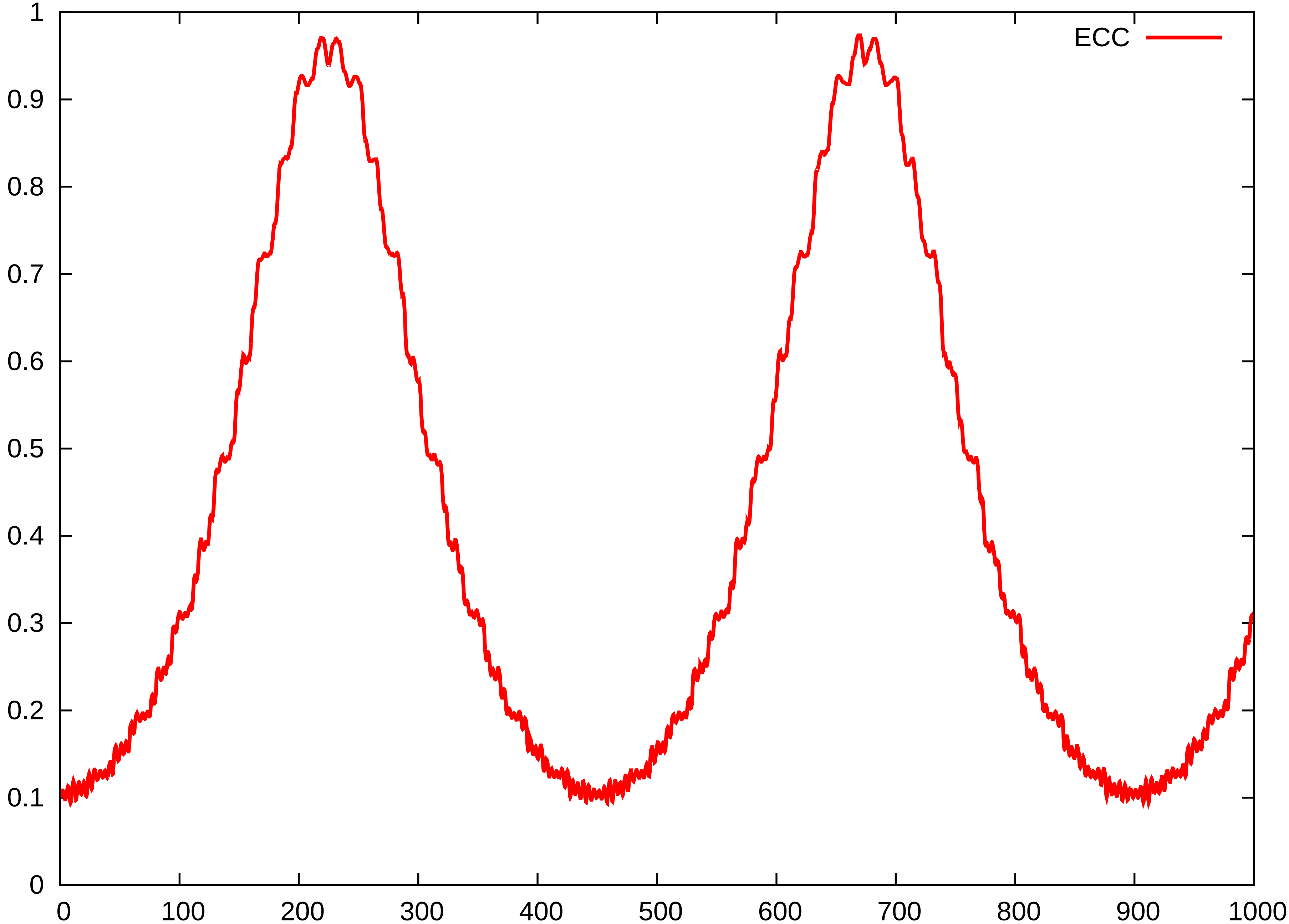
ETAI ETAR RS0 DTADJ DELTAT TCRIT QE RBAR ZMBAR

OPTIONS (50)

DTMIN RMIN ETAU ECLOSE GMIN GMAX

ALPHA BODY1 BODYN NBIN0 NHI0 ZMET EPOCH0 DTPLOT

Q 0 0 0 SMAX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0.6
0.5
0.4
0.3
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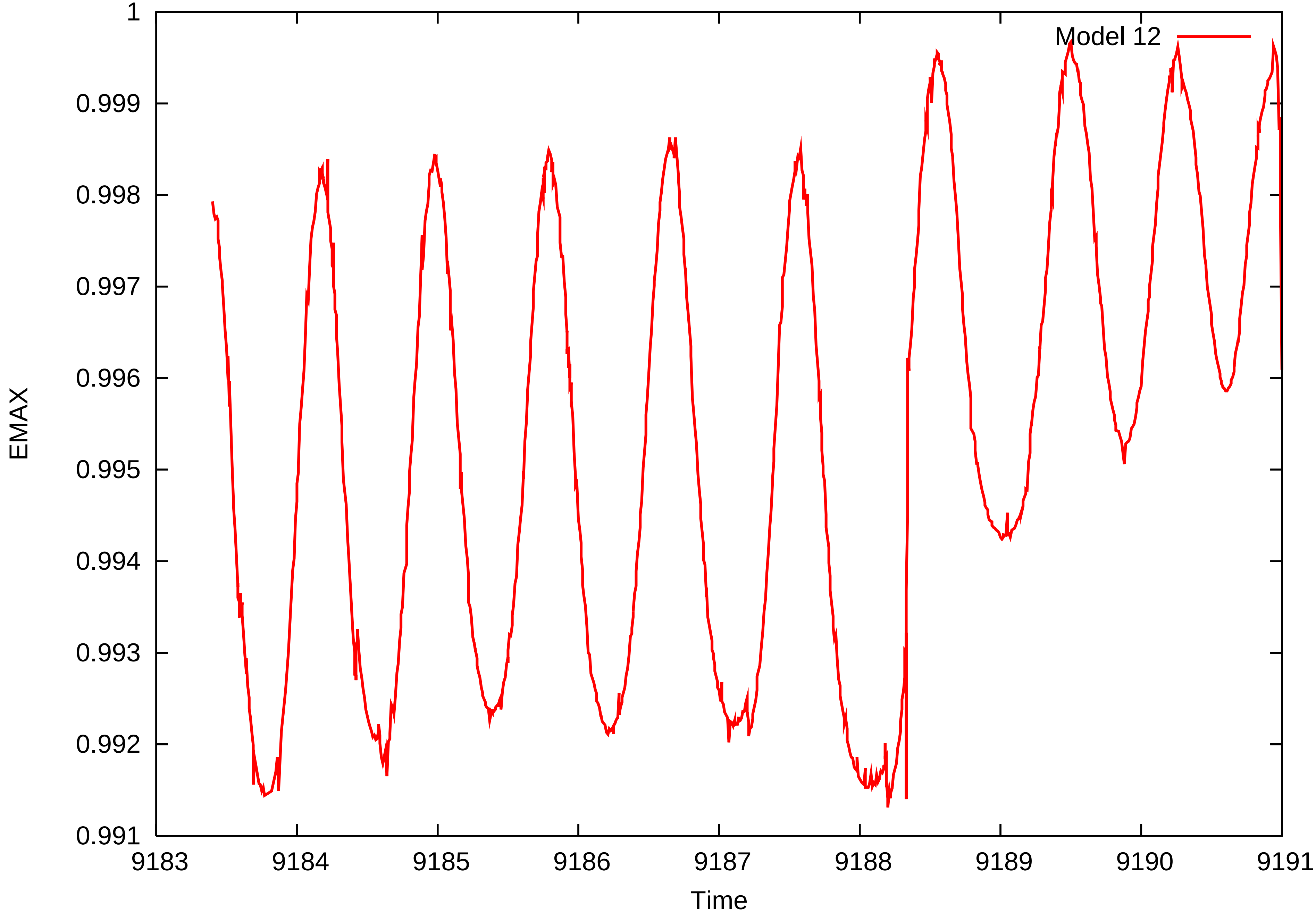
1000


Model 12
EMAX
Time
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0.998
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0.995
0.994
0.993
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0.991
9183
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Three-Body Stability Criterion

```
REAL*8 FUNCTION QSTAB(e,e_out, zi, m1, m2, m3)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 m1,m2,m3
*
* Three-body stability function (Valtonen, AAS 2015).
*
* Inner and outer eccentricity: e & e_out.
* Inclination in radians: zi.
* Masses: inner, m1  m2, outer, m3.
*
* Adopt 10,000 outer orbits for random walk time-scale.
ZN = 10000.0
zz = 1.0/6.0D0
*
F = 1.0 - 2.0*e/3.0*(1.0 - 0.5*e**2) - 0.3*cos(zi)*
&    (1.0 - 0.5*e + 2.0*cos(zi)*(1.0 - 2.5*e**1.5 - cos(zi)))
*
G = SQRT(m1/(m1 + m2))*(1.0 + m3/(m1 + m2))
*
QSTAB = 1.52*(SQRT(ZN)/(1.0 - e_out))**zz*(F*G)**0.3333

RETURN
END
```

Hierarchical Stability

Requirement	a_0 secularly constant
Kozai cycles	$e_{\rm max} = \left(1 - 5cos^2 i/3\right)^{1/2}$
Candidates	$\Delta t_{\rm cm} < \Delta t_{\rm cl},\ a_1(1 - e_1) > 3a_0$
Restrictions	$\mu_1 M_{123}/2a_1 > E_{\rm hard},\ \gamma_1 < 0.01$
Stability test	$f(a_0, a_1, e_0, e_1, \phi, m_1, m_2, m_3)$
Data structure	New KS, m_3 + inner c.m.
Merger table	$m_1, m_2, \mathbf{R}, \mathbf{V}, h, \mathbf{u}, \mathbf{u}', \mathcal{N}_g, \mathcal{N}_{cm}$
Initialization	New polynomials for KS and c.m.
Assessment	New check $R_{\rm apo} < P_{\rm crit}$
Mass loss	Update $h, \mathbf{u}, \mathbf{u}'$, stability check
Termination	$\gamma > 0.1$, $R > R_{\rm cl}$ or $\gamma > 0.25$
Re-initialize	Triple $\Rightarrow$ KS + m_3

Physical Collisions

Simple definition	$R_{\rm coll} = \frac{3}{4}(r_1^* + r_2^*)$
Two-body encounter	KS regularization
Pericentre condition	$R_0' \, R' < 0, \quad R < a$
Pericentre determination	$\Delta t_{\rm peri}$ from Kepler's equation
Predict $\mathbf{R}_{peri}$ or iterate	$d\tau_0 = \frac{\Delta t_{peri}}{R}$, Newton-Raphson
Implement collision	$m_{\rm cm} = m_1 + m_2, \quad r_{\rm cm} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}$
Mass loss	$\Delta m = f(K_1^*, K_2^*)$
Initialize single body	$\mathbf{F}_1, \, \dot{\mathbf{F}}_1, \, \Delta t_1$
Compact subsystem	$\dot{R} \simeq 0$ by iteration
Transformation	$\mathbf{Q}, \, \mathbf{P} \Rightarrow \mathbf{r}, \, \dot{\mathbf{r}}$
New chain construction	$N_{\rm ch} \Rightarrow N_{\rm ch} - 1, \quad E_{\rm coll} = E_{\rm ch} - \mathcal{V}$

Simulations

Initial conditions	$N = 1 \times 10^5,$ Kroupa IMF in $0.1 - 50 M_\odot$
Plummer & tides	$R_h = 2$ pc, $V^* = 10\, km/s,\ T^* = 0.2\,\mathrm{Myr}$
N-body units	$G = 1,\ \bar{m} = 1/N,\ \ E = -\frac{1}{4},\ \bar{v}^2 = \frac{1}{2}$
Hard binary	$\frac{m^2}{2a} \simeq \frac{1}{2}\bar{m}\bar{v}^2, \quad \Rightarrow a_{\mathrm{hard}} \simeq 2/N$
Super-hard	$-\frac{m^2}{2a} = E, \quad \Rightarrow a_0 = 2/N^2, \quad \text{or}\ \ 2 \times 10^{-10}$
Dynamics	$m_{\mathrm{bh}} \simeq 20\ M_\odot, \quad E_{\mathrm{b}} = 0.01E, \quad a \simeq 3 \times 10^{-5}$
Coalescence	$R_{\mathrm{co}} = \frac{8(m_1 + m_2)}{c^2}, \quad R_{\mathrm{co}} \simeq 7 \times 10^{-12}$
PN condition	$a(1 - e) \simeq 10^4 R_{\mathrm{co}}, \quad \Rightarrow e > 0.999$
Time-scale	$a = 3 \times 10^{-5},\ e = 0.999,\ m = 20\,\mathrm{M}_\odot,\ \tau_{\mathrm{co}} \simeq 200\,\mathrm{Myr}$
Einstein shift	$\Delta w = \frac{6\pi M}{ac^2(1 - e^2)} = 3 \times 10^{-4}$

Getting Started

1.	Download code	`nbody6.tar.gz`
2.	Unzip	`gunzip nbody6.tar.gz`
3.	Extract files	`tar xvf nbody6.tar`
4.	Check `params.h`	`NMAX, LMAX, KMAX, MMAX`
5.	Compile the code	`make nbody6`
6.	Create run directory	`mkdir Run`
7.	Run test input	`time nbody6 <input >output &`
8.	Profiling	Makefile with -O3 -pg
9.	Performance data	`gprof nbody6 gmon.out -p >OUT`