Dear students,

Thank you for your interest in my lectures on "Dynamics of black holes in dense stellar systems". Below, I have composed a few warm-up exercises in relation to the materials covered in these lectures which, I guess, you will enjoy too. Nevertheless, do not forget to familiarize yourselves to the fast-growing literature on this topic: the works cited in the slides will help you to get into it.

With best wishes,

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1. Explain quantitatively the concept of "negative specific heat" of a self-gravitating system.

   Hint: based on the virial theorem, derive a comprehensive formula for $dE/d\sigma$; $E =$ total energy, $\sigma =$ velocity dispersion (r.m.s. velocity w.r.t. the center of mass) of the system.

2. Consider clusters of $N = 100$, 5×10^4, 10^6, 10^8 all with half-mass radius $r_h = 2$ pc; for the most massive one take $r_h = 4$ pc. They are representatives of a medium-sized association, an open cluster, a globular cluster, and a galactic nuclear star cluster respectively. Evaluate the half-mass relaxation time for each of the representative case. How does this compare with the corresponding dynamical time for each case? How would you physically interpret the varying relaxation time to dynamical time ratios among these systems?

   Hint: for definiteness, assume a mean stellar mass of $0.3M_\odot$ in all cases.

3. Now consider a population of $30M_\odot$ stellar-mass black holes (BHs) retaining in each of the systems in Exercise 2. How long (say, in Myr), in each case, would it take for the BHs to segregate to the cluster center and form a "BH-core" (i.e., to reach the "first core collapse" phase)?

   Hint: you can use the relevant approximate formulae from the lecture slides.

4. Consider a binary black hole (BBH) comprising $30M_\odot$ BH members. Assuming that the proposed space-based LISA detector can detect BBHs of eccentricities $e \leq 0.7$, what would be the limits on the BBH's semi-major-axis, a, so that its peak gravitational-wave (GW) frequency, $f_{\rm GWp}$, is equal to 10^{-2} Hz (LISA would be most sensitive around this frequency)? What are the times that would be taken until the final merger via GW emission, for the BBHs at the extremities? How eccentric these BBHs have to be to undergo merger within 5 years (the intended lifetime of LISA mission)?

   Note: starting from the LISA frequency band, if a BBH can reach its final merger phase within the mission's lifetime, then the BBH would be visible both by LISA and LIGO detectors (assuming that it is sufficiently close).

5. Consider a particular BBH within the "BH-core" inside a globular cluster. For definiteness, take the average number density of the BHs within the BH-core to be $n = 10^4\ {\rm pc}^{-3}$ and their velocity dispersion to be $\sigma = 10\ {\rm km\ s}^{-1}$, as typical for a massive cluster. Assume the semi-major-axis of the BBH to be 100 AU and the masses of all BHs to be $10M_\odot$. How eccentric $(1-e < ?)$ the BBH has to be so that it undergoes GW coalescence in less than its own orbital time period (which event can be called a "GW-capture")? How eccentric it has to be to undergo a GW merger in between subsequent close binary-single encounters and to undergo a GW merger, within the Hubble time, being isolated? These would indicate the difference in binary configurations for GW-capture (say, during a resonant triple interaction), a general in-cluster GR merger (say, due to an eccentricity boost following a close binary-single encounter), and GR mergers after being dynamically ejected from a cluster.

   Hint: utilize the gravitational focusing formula to estimate the mean time between two close binary-single encounters.