Problems

Select one of the two problems according to your choice and solve it. For the task, you are free to use any common programming language which suits you at best. In case that you feel you need a piece of advice, do not hesitate to contact me at `dinnbier@sirrah.troja.mff.cuni.cz` .

Problem I: Initial conditions for a star cluster

Write a piece of code which generates the Plummer model in virial equilibrium. The Plummer parameter of the cluster is 0.5 pc and it contains 10^5 stars. All stars have the same mass of $1\,\mathrm{M}_\odot$. In the end check that you succeeded in generating the initial conditions by plotting the radial dependence of density ρ and velocity dispersion σ alongside the analytic values of these quantities for a Plummer model of this mass and radius.

Although all the necessary numerical tools were described at the lecture, I mention some literature below in case that something was unclear during the lecture. You can find the numerical recipe for generating the Plummer model in the Appendix of this paper `https://ui.adsabs.harvard.edu/abs/1974A%26A....37..183A/abstract`

I recommend you use the von Newmann rejection method for generating the velocity distribution of stars. Details of this method are described in the book Numerical Recipes by Teukolsky et al. (it is in their Chapter "Random numbers").

Problem II: Kozai-Lidov oscillations

One interesting phenomenon, which occurs in some of three body configurations is called Kozai-Lidov oscillations or Kozai-Lidov cycles. This typically occurs in stable configurations where an inner binary is orbited by a third body. For such a configuration to be stable, the semi-major axis of the third (outer) body should be at least 3 to 4 times larger than the semi-major axis of the inner binary. The inner binary also should have an appropriate choice of eccentricity and inclination (the inclination is calculated relatively to the orbit of the third body). The perturbing force of the outer body acting on the inner binary then causes periodic oscillations of the eccentricity and inclination of the third body. These oscillations occur on a time scale substantially longer than the orbital time-scale of the orbit of the third body. When the inclination reaches its maximum, the eccentricity is at its minimum (panel a in Fig. 1), and when inclination is at its minimum, the eccentricity reaches its maximum (panel b in Fig. 1).

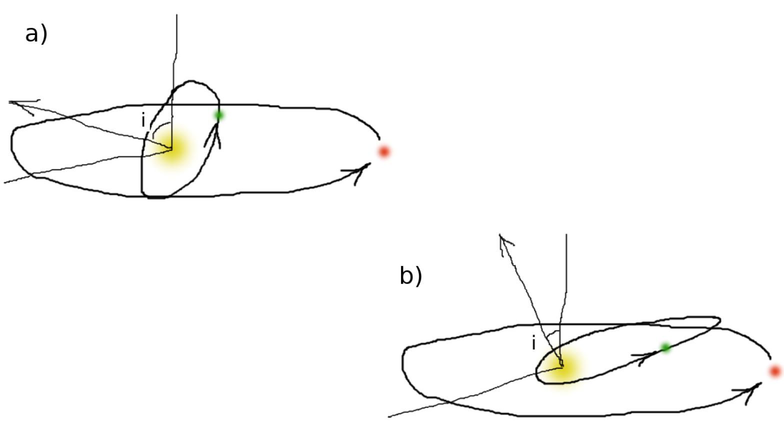

Figure 1: Cartoon illustrating the Kozai-Lidov oscillations. The inner binary is represented by the yellow and green stars (the green one is the test body), the outer (third) body is shown as the red star. Kozai-Lidov oscillations occur in the inner binary. FIGURE A): The inner binary is substantially inclined relative to the orbit of the outer body, but its eccentricity is small. FIGURE B): The inclination of the inner binary is smaller, but its orbit became eccentric.

The increase of eccentricity during the cycle means that the pericentre distance of the inner binary shrinks. Consequently, this opens the possibility of some interaction between the two bodies depending on their nature. For example, if the inner binary are black holes, the shrinkage of their pericentre distance causes gravitational radiation to be more efficient further dragging these bodies closer together, which can ultimately result in their coalescence. Thus, the Kozai-Lidov mechanism can drastically reduce time-scale for coalescence between two black holes. Less exotic systems subjected to the same mechanism are hot Jupiters (if perturbed by another more distant planet or star), which orbit other stars, and triple or multiple stellar systems.

Your task is to write a second-order leap-frog integrator which can handle three mutually interacting bodies [1] . I recommend using kick-drift-kick flavour of the integrator. Then initialise a three body problem with the following initial conditions: Consider a star of mass of $1\,\mathrm{M}_\odot$, which is at the rest and at the centre of the system. There is an M dwarf of mass $0.1\,\mathrm{M}_\odot$ orbiting the $1\,\mathrm{M}_\odot$ star on a circular orbit at a distance of 6 AU. There is a test particle of mass 1 gram orbiting the $1\,\mathrm{M}_\odot$ star at a distance of 1 AU. The orbit of the test particle is initially mildly eccentric ($\epsilon = 0.1$)

[1] If you look up the equations for the leap-frog integrator from the textbook Binney & Tremaine 2007, please note that there is a mistake in their equations (3.166a) and (3.166b). For the correct form of the equation, take a look at their errata, or at the slides from my talk.

and highly inclined ($i = 80°$) relative to the orbit of the M dwarf. Integrate the trajectories of the three bodies for several thousands orbital times of the inner body, and plot the time dependence of the following quantities:

1. Check the conservation of the mechanical energy in the system (i.e. the sum of the potential and kinetic energy) by plotting the time dependence of the relative energy error.
2. What is the maximum relative energy error in your calculations? Does the error systematically increase with time?
3. The eccentricity of the inner binary [2]
4. Inclination i of the orbit of the test particle (relative to the orbital plane of the M dwarf).
5. What is the period of the oscillation of ϵ or i?

[2] Useful formula: The semi-major axis a of the inner binary can be calculated from the energy conservation, i.e.

$$-\frac{Gm_1m_2}{2a} = \frac{1}{2}m_1v^2 - \frac{Gm_1m_2}{r}, \tag{1}$$

where m_1, m_2 is the mass of the components forming the inner binary, v their relative velocity, r their relative distance, and G the gravitational constant. Than the eccentricity ϵ can be calculated from the following formula

$$\epsilon^2 = (1 - r/a)^2 + \frac{(\mathbf{r} \cdot \dot{\mathbf{r}})^2}{G(m_1 + m_2)a}, \tag{2}$$

where $\mathbf{r}$ and $\dot{\mathbf{r}}$ is the relative distance and velocity between the two stars. Obviously, $r = |\mathbf{r}|$ and $v = |\dot{\mathbf{r}}|$.