Newtonian Dynamics in Astrophysics

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Prague, December 2022

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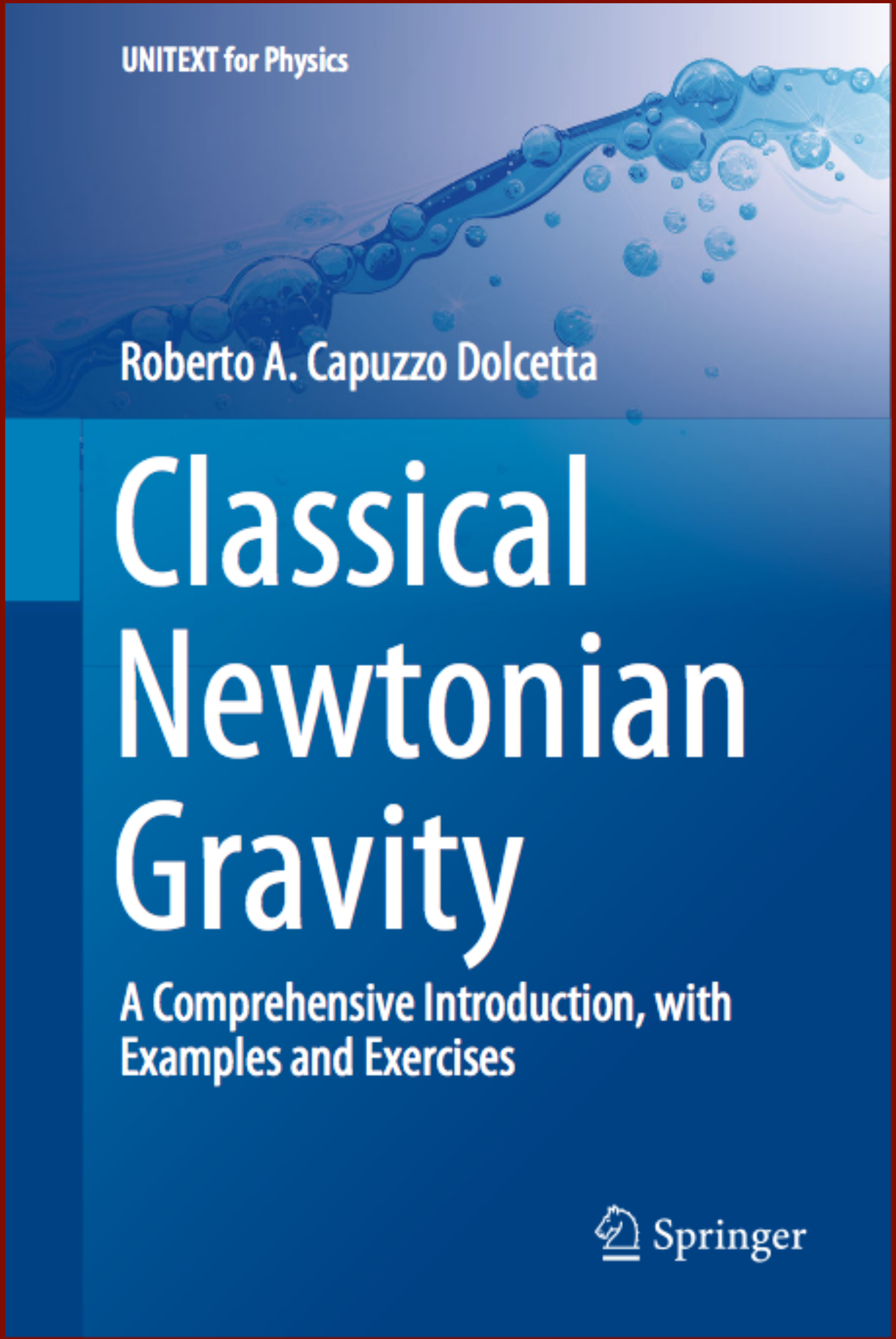
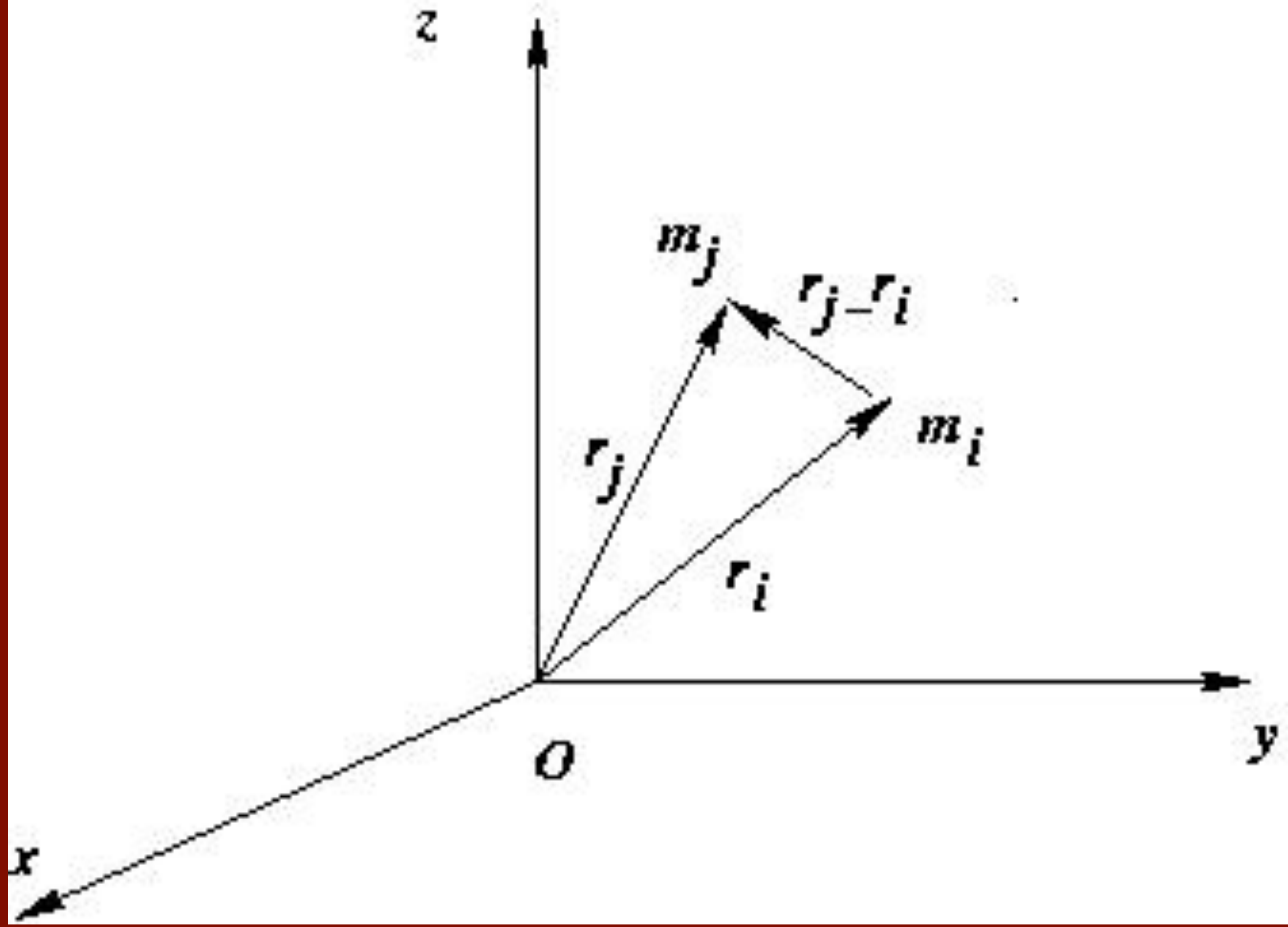


2nd Lecture, Dec. 7, 2022

The *N*-body classic gravitational problem

Statement of the problem



$$\ddot{\mathbf{r}}_i = -G \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{m_j}{|\mathbf{r}_i - \mathbf{r}_j|^3} \left(\mathbf{r}_i - \mathbf{r}_j\right)$$

$$\mathbf{r}_i(0) = \mathbf{r}_{i0}$$

$$\dot{\mathbf{r}}_i(0) = \dot{\mathbf{r}}_{i0}$$

Explicit solutions only for N=2. Under simplifying conditions for N=3.
As (unusable) series for $N \geq 3$ and $L > 0$.

The system can be transformed into a system of 6*N* *1st* order ODEs letting $\mathbf{v}_i = \dot{\mathbf{r}}_i$, $i=1,2,\ldots,N$:

$$\begin{cases} \dot{\mathbf{r}}_i = \mathbf{v}_i, \\ \dot{\mathbf{v}}_i = G \sum\limits_{\substack{j=1 \\ j\neq i}}^{N} m_j \frac{\mathbf{r}_j - \mathbf{r}_i}{r_{ij}^3}, \\ \mathbf{r}_i(0) = \mathbf{r}_{i0}, \\ \mathbf{v}_i(0) = \dot{\mathbf{r}}_{i0}. \end{cases}$$

In any case the system:

- is of *complexity* O(N^2);
- is far from *linearity*;
- is poorly *constrained* in phase-space —> chaotic (ergodic behavior) .

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10 constants of motion, independent of N

$$E = \frac{1}{2}\sum_{i=1}^{N} m_i v_i^2 - \frac{1}{2}\sum_{(i,j)=1, i\neq j}^{N} G\frac{m_i m_j}{r_{ij}} = E_0,$$ *Energy first integral*

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \wedge m_i \dot{\mathbf{r}}_i = \mathbf{L}_0,$$ *Angular momentum first integral*

$$\mathbf{Q} = \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i = \mathbf{Q}_0,$$ *Momentum first integral*

$$\mathbf{r}_{com} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{M} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_{i0}}{M} + \frac{\mathbf{Q}_0 (t - t_0)}{M}.$$ *Center of mass constant*

Explicit solutions only for N=2. Under simplifying conditions for N=3. As (unusable) series for $N \geq 3$ and $L > 0$.

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Assignement

Write from <u>scratch</u> a computer code to integrate numerically the time evolution of Newtonian *N*-body system for given initial conditions.

- The "stars" of the system are *point-like* and have masses over a range $m_{min} \leq m \leq m_{max}$, distributed according a given IMF, *φ(m)* ;
- the total mass would be $M = N <m>$, where $<m>$ is the average mass, as coming from *φ(m)*: $\langle m \rangle = \frac{\int_0^\infty \varphi(m) m dm}{N}$;

- initial conditions for positions and velocities $(\mathbf{r}_{i0}, \mathbf{v}_{i0})$ must be sampled by chosen *sp* *velocity* distributions in spherical symmetry;
- in a second step, an external gravitational potential should be included.

Important: DISCUSS and INTERPRET RESULTS!

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An historical note: the king Oscar II prize

In 1885 Gösta Mittag-Leffler proposed king Oscar II of Sweden to award a prize of a gold medal and 2500 kr for a solution to an unsolved problem among 4 proposed by the committee (Mittag-Leffler, Hermite and Weierstrass).

The first question (which attracted attention of 5 out of 12 applicants) was:

For a system of arbitrarily many mass points that attract each other according to Newton's law, assuming that no two points ever collide, find a series expansion of the coordinates of each point in known functions of time converging uniformly for any period of time.

Answer due within June 1 1888

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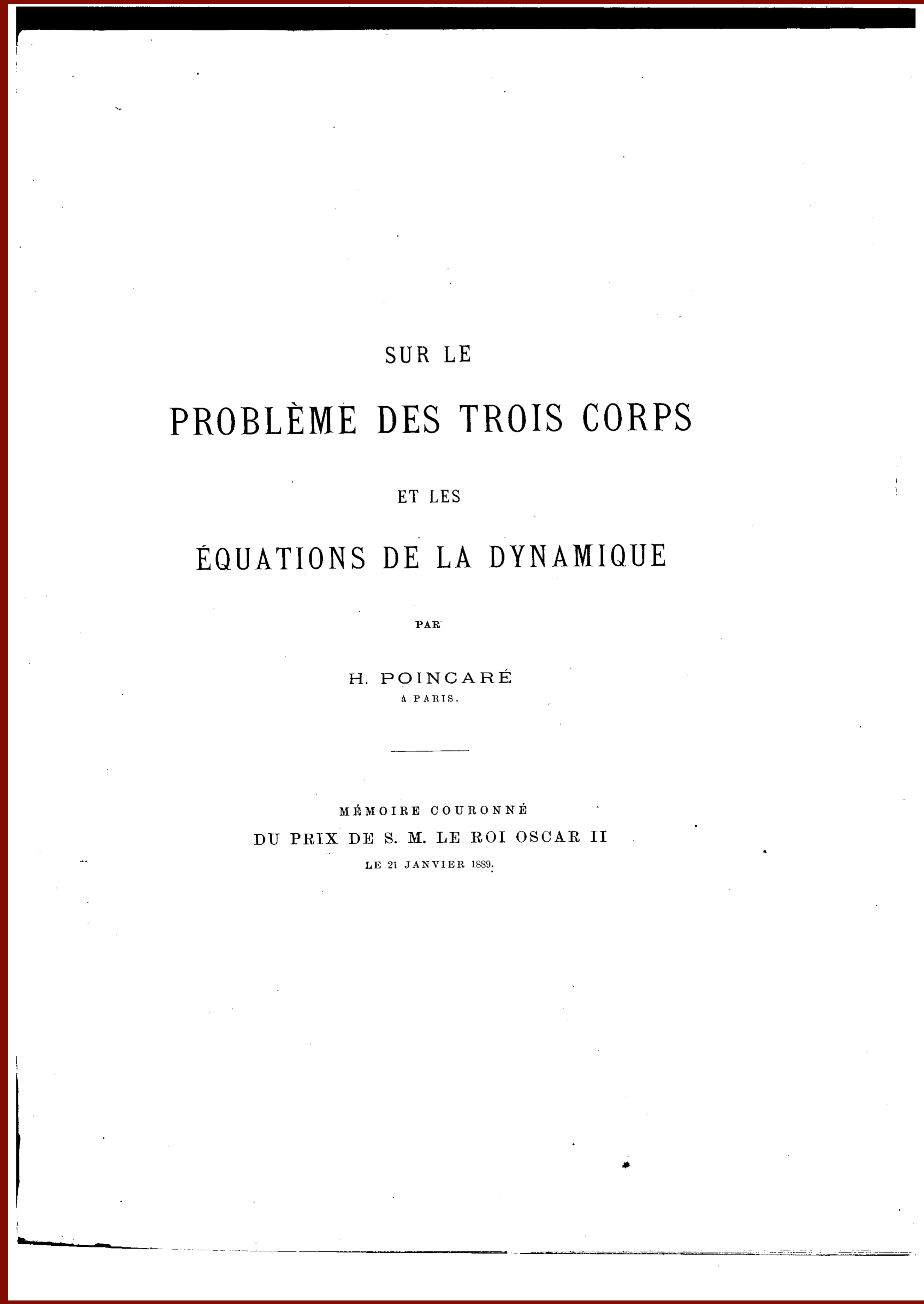
Henry Poincaré was awarded, although there was an error
in his *memoire* on the three body problem. But his paper gave rise to the concept o
deterministic chaos.

Henri Poincaré
1854-1912



SUR LE

PROBLÈME DES TROIS CORPS

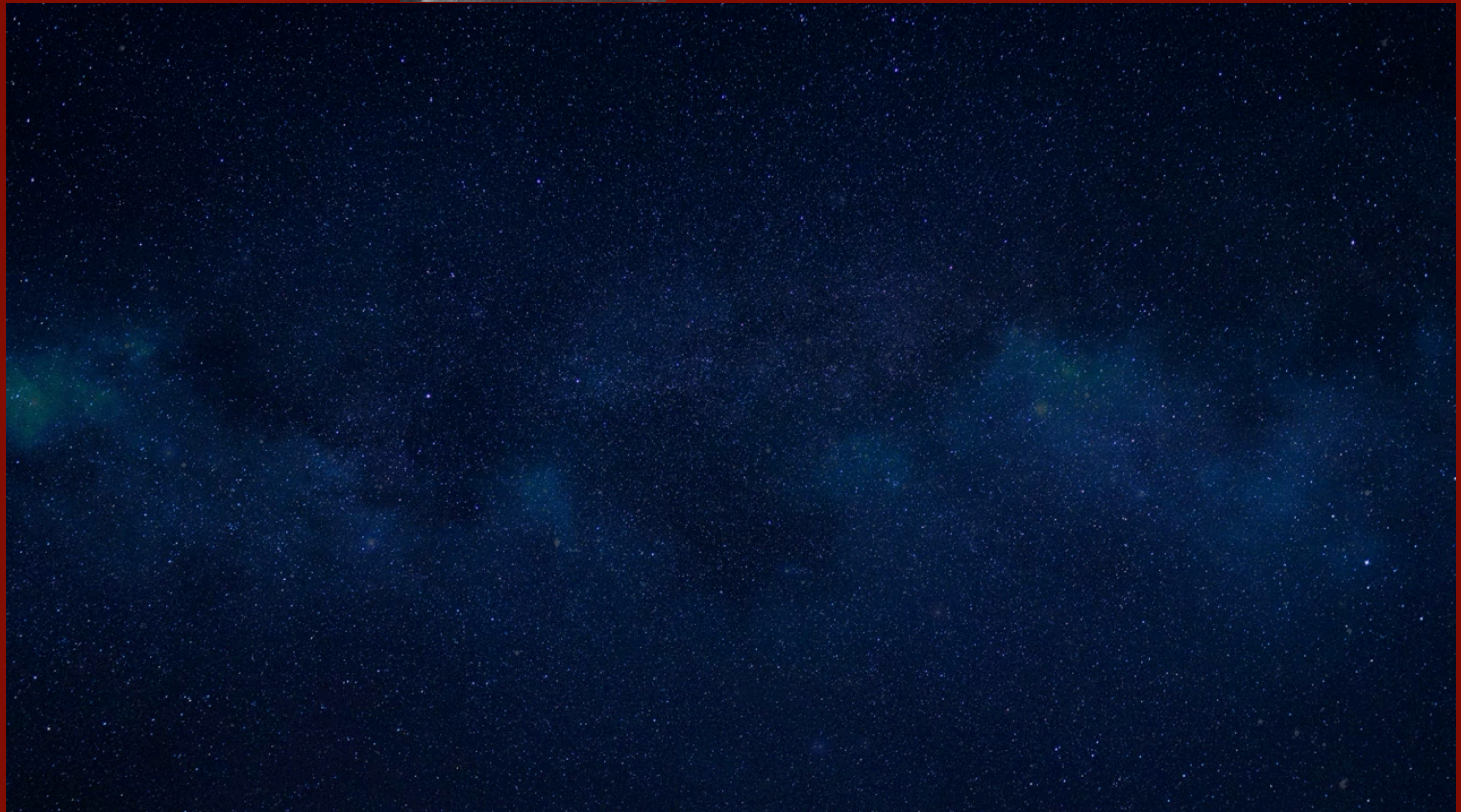
ET LES

ÉQUATIONS DE LA DYNAMIQUE

PAR

H. POINCARÉ
À PARIS.

MÉMOIRE COURONNÉ
DU PRIX DE S. M. LE ROI OSCAR II
LE 21 JANVIER 1889.



Sundman in 1912 showed that, for $N = 3$, there exists a series solution in powers o
convergent for all t, except for initial data which correspond to zero angular mome
(unfortunately, the prize has already gone to Poincarè).

This result was generalized to any N only in 1991 by Q.Wang.
Anyway, the power series solutions are so slow in convergence to be useless for pr
use.

The (scalar) virial theorem

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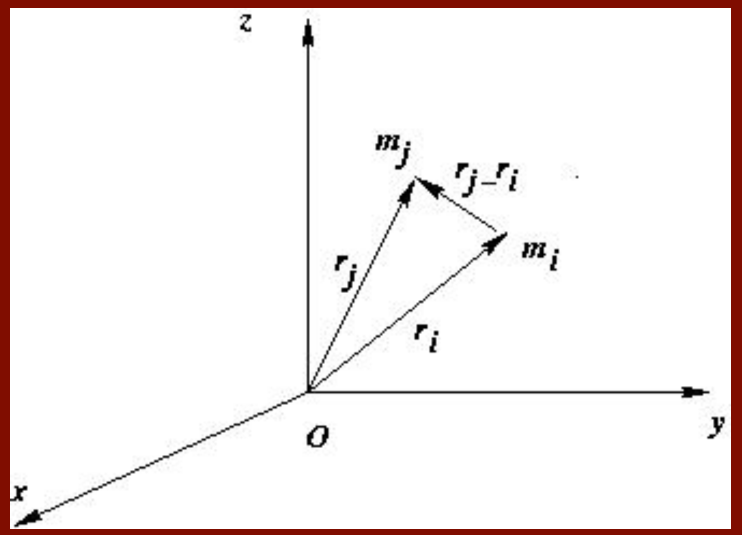
Inertia moment $I = \sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \mathbf{r}_i \longrightarrow \frac{1}{2}\ddot{I} = 2T + \sum_{i=1}^{N} \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i \,,$



Clausius *virial*

$$\ddot{\mathbf{r}}_i = \nabla_i U + \frac{\mathbf{F}_i^{ext}}{m_i}, \quad U(r_1, \cdots r_N) = \frac{G}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{m_i m_j}{|r_i - r_j|}, \quad \text{if } \mathbf{F}_i^{ext} = 0, \text{ then}$$

$$\sum_{i=1}^{N} \mathbf{r}_i \cdot \nabla_i U = \sum_{i=1}^{N} \left(x_i \frac{\partial U}{\partial x_i} + y_i \frac{\partial U}{\partial y_i} + z_i \frac{\partial U}{\partial z_i} \right) = nU,$$

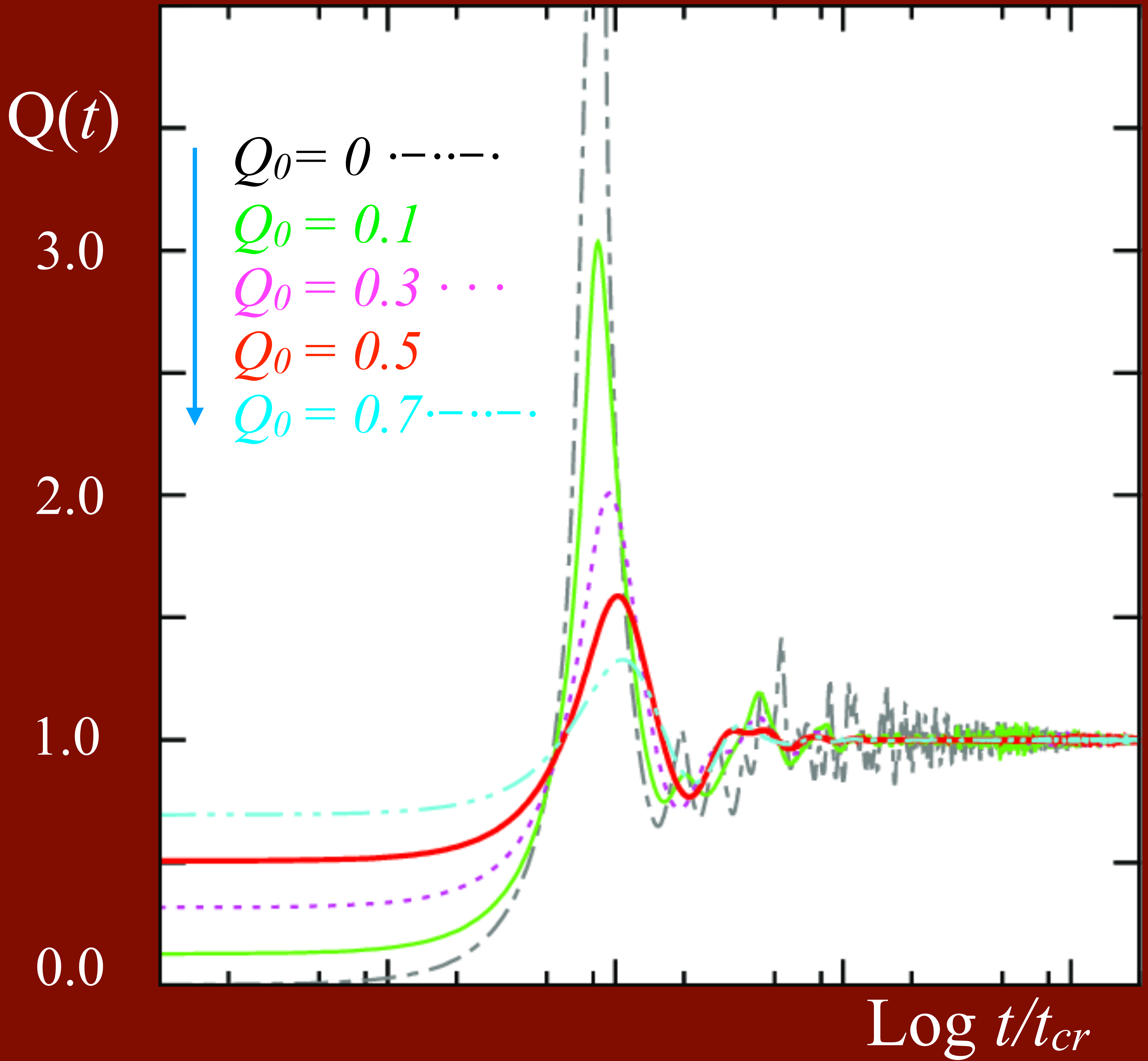
$$\frac{1}{2}\ddot{I} = 2T + nU = \frac{1}{2}\ddot{I} = 2T - U = 2T + \Omega = T + E,$$

$$\frac{1}{2}\frac{\ddot{I}}{|\Omega|} = \frac{2T}{|\Omega|} - 1 \equiv Q - 1 \Longleftrightarrow Q = \frac{1}{2}\frac{\ddot{I}}{|\Omega|} + 1$$

so that $Q = 1$ at *virial equilibrium*, and $Q = 2\left(\frac{E}{|\Omega|} + 1\right) \leq 2$ for bound systems $(E \leq 0)$.

So: $0 \leq Q \leq 2$ and $Q = 0$ at *virial equilibrium*

If $\ddot{I} = 0$ *then* $\frac{dE}{dT} < 0 \Longleftrightarrow$ negative specific heat!



Q(t)
3.0
2.0
1.0
0.0
$Q_0 = 0$
$Q_0 = 0.1$
$Q_0 = 0.3$
$Q_0 = 0.5$
$Q_0 = 0.7$
Log t/t_{cr}

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And if external forces are present?

$$\Downarrow$$

$$\frac{1}{2}\ddot{I} = 2T + \sum_{i=1}^{N} \mathbf{r}_i \cdot \ddot{\mathbf{r}}_i = 2T + \Omega - \sum_{i=1}^{N} \frac{F_i^{ext}}{m_i} r_i, \quad \mathbf{F}_i^{ext} = -F_i^{ext} \frac{\mathbf{r}_i}{r_i}$$

$$Q = \frac{1}{2} \frac{\ddot{I}}{|\Omega|} + 1 + \sum_{i=1}^{N} \frac{F_i^{ext}}{m_i} r_i > 1 \text{ at equilibrium } (\ddot{I} = 0)$$

$E > 0 \Longrightarrow$ system spatially unstable

$$\frac{1}{2}\ddot{I} = E + T \geq E > 0, \Longrightarrow I(t) \geq Et^2 + \dot{I}(0)t + I(0)$$

$$E>0 \quad \Downarrow$$

$$\lim_{t\to\infty} I(t) = +\infty$$

that means the system is indeed *unstable*, in what exists at least an *escaping particle*, for which $r_i(t) \longrightarrow \infty$

Note: this of course does not imply that $E < 0$ corresponds to stable system: think t holes!

Simply implies that if the systems is spatially stable its $E < 0$

Self gravitating systems are difficult to study due to the *double divergence* of $U_{ij} \propto 1/r_{ij}$

$\Downarrow$

1) *IR divergence* (U_{ij} never vanishes) $\Rightarrow O(N^2)$

2) *UV divergence* $\left(\lim_{r_{ij}\to 0} U_{ij} = \infty\right)$ $\Rightarrow \Delta t \to 0$

UV divergence $(\lim_{r_{ij}\to 0} U_{ij} = \infty)$

"granularity" i.e. fluctuations over the mean field.

Distinction between *collisionless* and *collisional* systems.

Collisionless: fluctuations <u>irrelevant</u>
Collisional: fluctuations <u>relevant</u>

fluc. over mean field $\sim \frac{\sqrt{N}}{N}$ $\Rightarrow$ galaxies are *<u>collisionless</u>*

open clusters are *<u>collisional</u>*

Self-gravitating systems: from *small* to *large N*

OC and GC: an *Intermediate N* body problem ($10^2 \div 10^7$)

$t_{trel} <$ age $\Rightarrow$ collisional; $t_{cross} << t_{trel} <$ age $\Rightarrow$ sec. collisional

The multiplicity of *time scales* requires

individual time stepping

constant Δt → wrong!

variable Δt → correct!