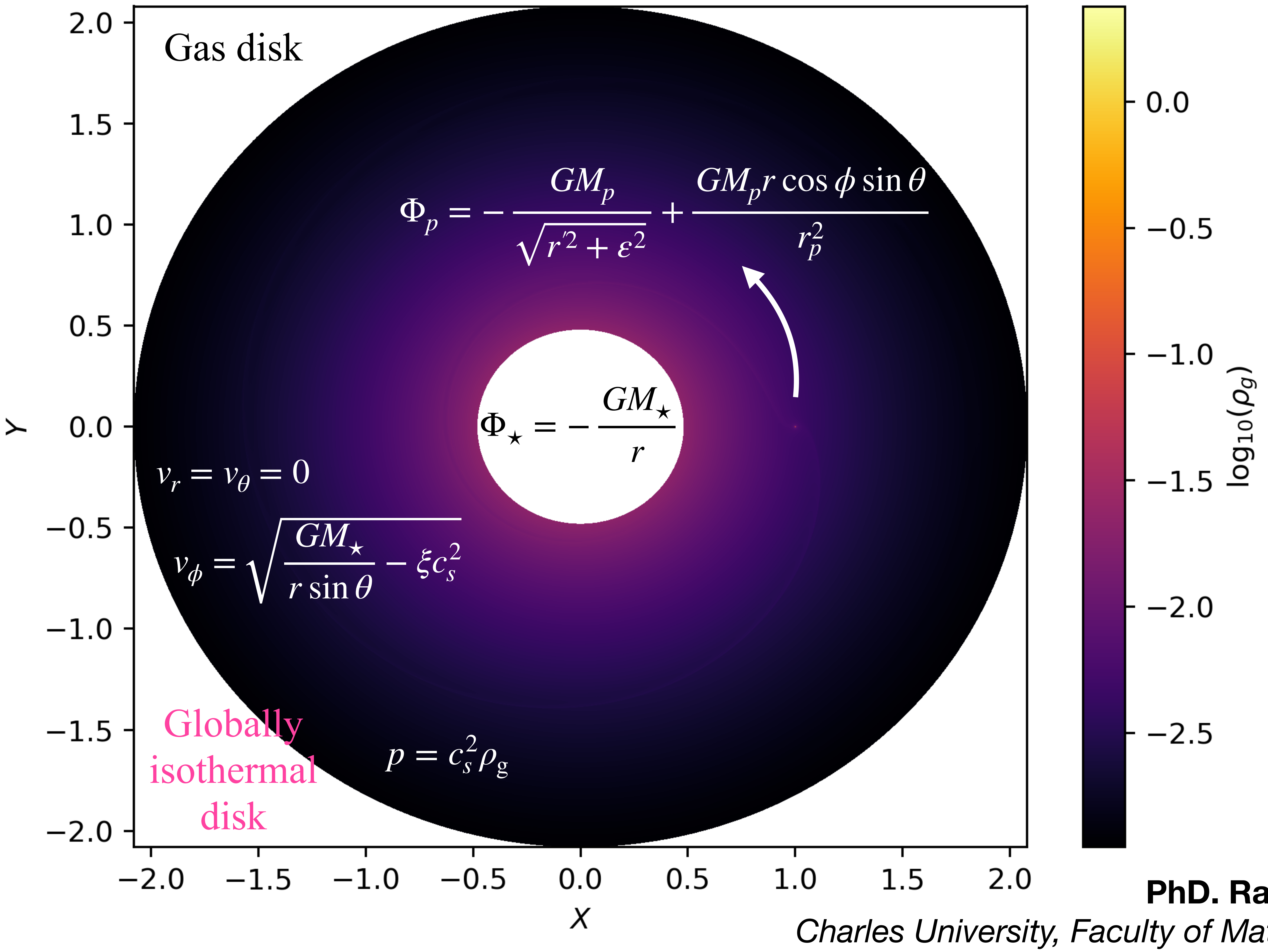
¨Overview of recent advances in planetary migration: from theoretical models to high-resolution 3D multi-fluid simulations¨



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We assume that the sound speed is a power law of the spherical radius:

$$c_s^2(r) = \left(c_s^0\right)^2 \left(\frac{r}{r_0}\right)^{-\beta}, \tag{1}$$

where r_0 is an arbitrary radius at which the sound speed is c_s^0. Such disks are often said to be locally isothermal. The aspect ratio has the radial dependence

$$h(r) = \frac{c_s(r)}{v_K(r)} \propto r^{(1-\beta)/2}, \tag{2}$$

where $v_K(r) = \sqrt{GM_\star/r}$ is the circular Keplerian velocity at distance r from the central mass. We call the flaring index the exponent f of the power law given by Equation (2):

$$f = \frac{1-\beta}{2}. \tag{3}$$

The equations that determine the rotational and vertical equilibria of the disk are respectively, in spherical coordinates:

$$-\frac{\partial_r\left(\rho_0 c_s^2\right)}{\rho_0} + \frac{v_\phi^2}{r} - \frac{GM_\star}{r^2} = 0 \quad (4)$$

and

$$-\frac{1}{r}\frac{\partial_\theta\left(\rho_0 c_s^2\right)}{\rho_0} + \frac{v_\phi^2}{r}\cot\theta = 0. \quad (5)$$

If we denote $L = \log(\rho_0/\rho_{00})$, $m = v_\phi^2/c_s^2$, $u = -\log(\sin\theta)$, $v = \log(r/r_0)$, and $K = GM_\star/[c_s(r_0)^2 r_0]$, we can transform Equation (5) into

$$\partial_u L + m = 0 \quad (6)$$

and Equation (4) into

$$-\beta + \partial_v L - m + K\exp(-2fv) = 0, \quad (7)$$

where we have made use of the assumption that the sound speed depends only on the spherical radius. Differentiating Equation (6) with respect to v and Equation (7) with respect to u, we are led to

$$\partial_v m + \partial_u m = 0, \tag{8}$$

from which we infer

$$\partial^k_{u^k} m = (-1)^k \partial^k_{v^k} m. \tag{9}$$

The rotational equilibrium in the midplane reads, from Equation (4),

$$m(u = 0, v) = -\beta - \xi + K \exp(-2fv), \tag{10}$$

hence, for any $k \geqslant 1$, we have in the midplane ($u = 0$)

$$\partial^k_{v^k} m = (-2f)^k K \exp(-2fv), \tag{11}$$

so that, by virtue of Equation (9), we have, also in the midplane,

$$\partial^k_{u^k} m = (2f)^k K \exp(-2fv), \tag{12}$$

from which we can reconstruct the value of m at an arbitrary height above the midplane:

$$\begin{aligned} m(u, v) &= [\exp(2fu) - 1]K \exp(-2fv) + m(u = 0, v) \\ &= -\beta - \xi + K \exp[2f(u - v)], \end{aligned} \tag{13}$$

which specifies the field of rotational velocity. The density field is found by integrating Equation (6), which yields

$$L = L_{\mathrm{eq}} + (\beta + \xi)u - K\frac{e^{-2fv}}{2f}[\exp(2fu) - 1], \tag{14}$$

where the subscript eq denotes the midplane value. Using the more conventional notation, Equations (13) and (14) read respectively

$$v_\phi(r, \theta) = v_K(r)\left[(\sin\theta)^{-2f} - (\beta + \xi)h^2\right]^{1/2}, \tag{15}$$

where $v_K(r)$ is the circular Keplerian velocity at distance r from the central mass, and

$$\rho_0(r, \theta) = \rho_{\mathrm{eq}}(r)(\sin\theta)^{-\beta-\xi} \exp\left[h^{-2}\left(1 - \sin^{-2f}\theta\right)/2f\right]. \tag{16}$$

For a "flat" disk, in which the temperature is inversely proportional to the radius ($\beta = 1$ and $f = 0$), the integration of Equation (6) eventually yields

$$\rho_0(r, \theta) = \rho_{\rm eq} (\sin \theta)^{-\beta-\xi+h^{-2}}. \tag{17}$$

For globally isothermal disks, Equations (15) and (16) can be recast respectively as

$$\rho(r, \theta) = \rho_{\rm eq} \sin^{-\xi} \theta \exp\left[h^{-2}\left(1 - \frac{1}{\sin \theta}\right)\right] \tag{18}$$

$$v_\phi^2(r, \theta) = \frac{GM_\star}{r \sin \theta} - \xi c_s^2 = \frac{GM_\star}{R} - \xi c_s^2. \tag{19}$$

The rotational velocity is therefore independent of the altitude at a given cylindrical radius in globally isothermal disks.

Finally, for $z/R \ll 1$, we have $u \approx \frac{1}{2}(z/R)^2$, hence Equation (14) can be recast in the following approximate form, when $fu \ll 1$:

$$L \approx L_{\rm eq} - \frac{1}{2} h(r_0)^{-2} \left(\frac{r}{r_0} \right)^{-2f} \left(\frac{z}{R} \right)^2, \tag{20}$$

where use has been made of the relationship $h^{-2} \gg |\xi + \beta|$. As a consequence, we recover the well-known approximation

$$\rho_0(z) \approx \rho_{\rm eq} \exp\left(-z^2/2H^2\right), \tag{21}$$

from which we can infer the relationships

$$\Sigma_0(r) = \sqrt{2\pi}\, \rho_{\rm eq} H \tag{22}$$

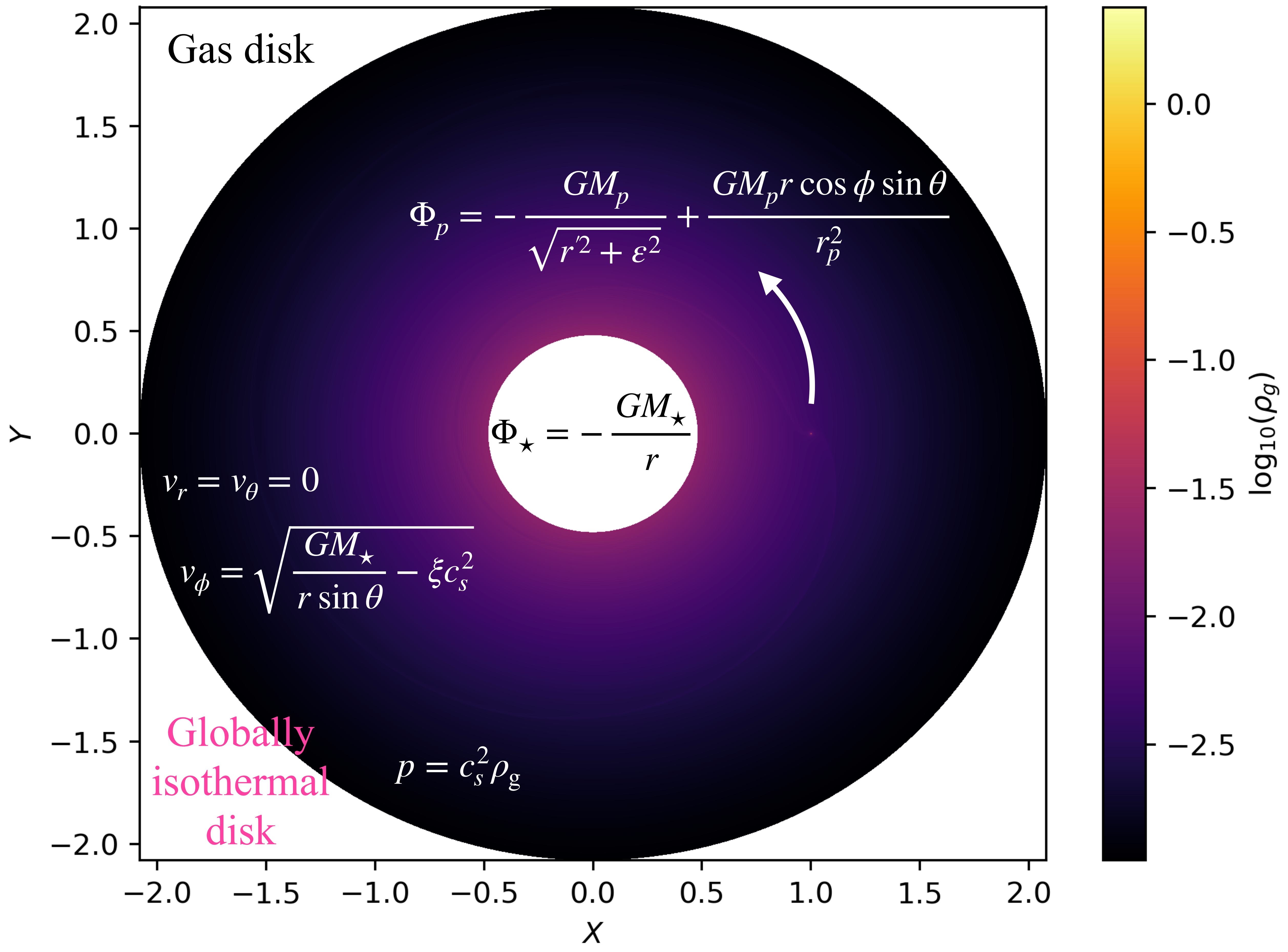
and

$$\alpha = \xi - 1 - f. \tag{23}$$

Planet dusty disk model

$$\partial_t \rho_g + \nabla \cdot (\rho_g \mathbf{v}) = 0$$

$$\partial_t (\rho_g \mathbf{v}) + \nabla \cdot (\rho_g \mathbf{v} \otimes \mathbf{v} + p\mathbf{I}) = -\nabla p - \rho_g \nabla \Phi - \mathbf{f}_d$$



We can perform 3D high resolution
$(N_r, N_\theta, N_\phi) = (H/100, H/100, H/100)$
multi fluid simulations using FARGO3D code

Dust disk $\quad \partial_t \rho_d + \nabla \cdot (\rho_d \mathbf{u}) = 0$

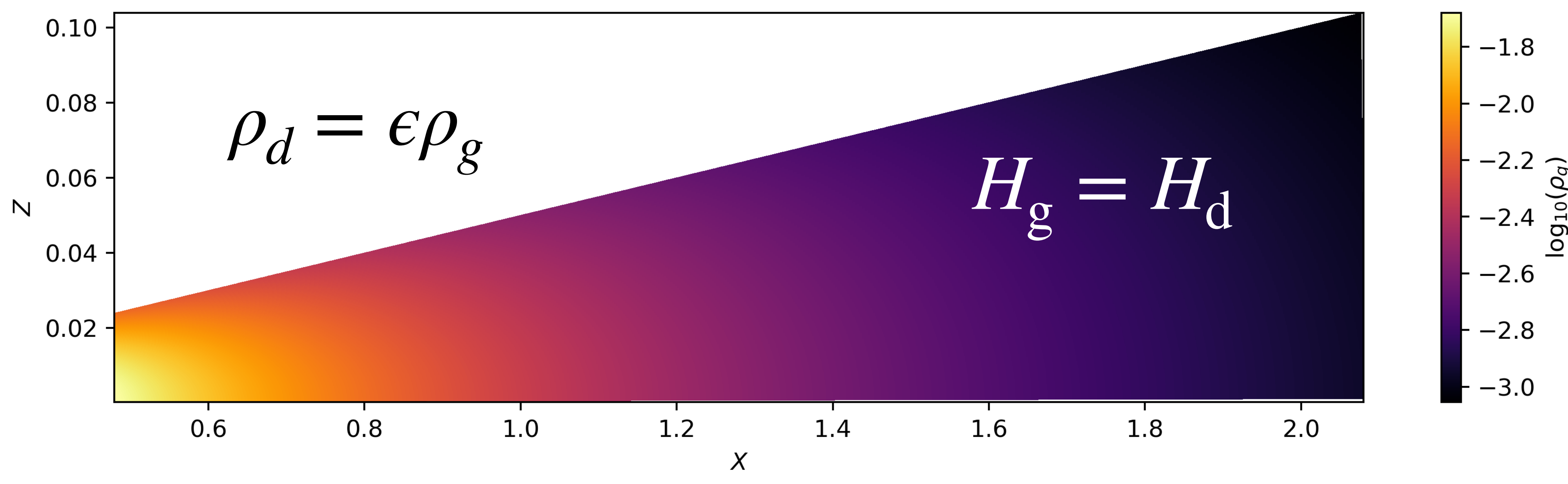
$$\partial_t (\rho_d \mathbf{u}) + \nabla \cdot (\rho_d \mathbf{u} \otimes \mathbf{u}) = -\rho_d \nabla \Phi + \mathbf{f}_d$$

$$St = \frac{\pi}{2} \frac{s_d \rho_s}{\Sigma_g}, \qquad \Sigma_g(r) = \Sigma_0 \left(\frac{r}{r_p}\right)^{-\alpha},$$

Drag force

$$\mathbf{f}_d = \Omega_K St^{-1} (\mathbf{v} - \mathbf{u}).$$

Dust feedback on the gas is included!



Note: we also consider thin layered dusty disks with $H_d = \sqrt{\frac{\alpha_d}{\alpha_d + St}} H_g$.