Homework Set: Chemical Enrichment and the IMF
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Problem 1

1. Summarize 5 to 6 major astrophysical sources of chemical enrichment. For each source, state which broad element groups it contributes most strongly to (for example: Be and B, alpha elements, odd-Z elements, light elements, heavy elements). State when or where each source tends to be most important, (for example, early versus late times after star formation).

2. Identify the main nuclear physics mechanisms that synthesize the elements and verify that most broad categories were already discussed in the B^2FH paper.

Problem 2

Using Table 1 of Kroupa+2013, reproduce the quoted **number fractions** and **mass fractions** for each IMF mass interval. Treat this as an exercise in computing integrals of a piecewise power-law IMF.

Then explore how these fractions change if the IMF shape is modified:

1. **Top-heavy IMF:** repeat the calculation with a flatter high-mass slope, for example $\alpha_3 = 1.6$ and $\alpha_3 = 0.6$ (leave the lower-mass slopes unchanged).

2. **Bottom-heavy IMF:** repeat the calculation with a steeper intermediate-mass slope, for example set $\alpha_1 = 2.3$ and $\alpha_2 = \alpha_1 + 1$ (and keep $\alpha_3 = 2.3$ unchanged).

You may either write your own code or use the provided notebook `example_Prague.ipynb`, which calls the pyIGIMF package. Report your results and comment a short interpretation: for each modified IMF (top-heavy and bottom-heavy), state which IMF intervals contain a **larger or smaller fraction of the total number of stars** and which intervals contain a **larger**

or smaller fraction of the total stellar mass, compared to the canonical IMF values in Table 1 of Kroupa+2013.

Problem 3

For an individual main-sequence star, a rough bolometric mass-to-light ratio follows from a piecewise power-law mass–luminosity relation. Let M be the stellar mass, $L_{\rm bol}(M)$ the bolometric luminosity, and $M_\odot$ and $L_\odot$ the solar mass and luminosity. Take the bolometric mass-to-light ratio,

$$\left(\frac{M}{L}\right)_{\rm bol}(M) = \frac{M}{L_{\rm bol}(M)} \tag{1}$$

to be:

$$\left(\frac{M}{L}\right)_{\rm bol}(M) \approx \frac{M_\odot}{L_\odot} \times \begin{cases} \frac{1}{0.23}\left(\frac{M}{M_\odot}\right)^{-1.3}, & 0.08 \le \frac{M}{M_\odot} < 0.421, \\ \left(\frac{M}{M_\odot}\right)^{-3}, & 0.421 \le \frac{M}{M_\odot} < 1.96, \\ \frac{1}{1.4}\left(\frac{M}{M_\odot}\right)^{-2.5}, & 1.96 \le \frac{M}{M_\odot} < 55.41, \\ \frac{1}{32000}, & 55.41 \le \frac{M}{M_\odot} \le 100. \end{cases} \tag{2}$$

Adapted from Yan+2017, their Eq. 1. This is intended as an approximate scaling for main-sequence stars only[1].

Plot the mass-to-light ratio as a function of stellar mass.

1. What trend does the plot show?
2. How do the luminosities of low-mass stars and massive stars compare?
3. How many orders of magnitudes are spanned by stellar mass and stellar luminosity?

Consider a single stellar population (SSP) formed in an instantaneous burst with total initial stellar mass $M_{\rm SSP}$ and an IMF $\xi(M)$ (number of stars per unit mass). Assume the IMF is defined over the mass range

$$M_{\rm min} = 0.08\,M_\odot \le M \le M_{\rm max} = 150\,M_\odot. \tag{3}$$

Use the *main-sequence* bolometric mass–luminosity relation given above (Yan et al.), and extend its highest-mass branch to $150\,M_\odot$ using the same scaling for $M \ge 55.41\,M_\odot$.

[1] Post-main-sequence phases can dominate $L_{\rm bol}$ at fixed M, so $(M/L)_{\rm bol}$ for an evolved star can differ by orders of magnitude

Define the total bolometric luminosity of the SSP as

$$L_{\rm SSP} \equiv \int_{M_{\rm min}}^{M_{\rm max}} L_{\rm bol}(M)\, \xi(M)\, {\rm d}M, \tag{4}$$

and normalize the IMF by the condition

$$M_{\rm SSP} \equiv \int_{M_{\rm min}}^{M_{\rm max}} M\, \xi(M)\, {\rm d}M. \tag{5}$$

You will compare two IMFs: (i) a canonical IMF (Kroupa, 2001), and (ii) a top-heavy IMF (e.g., $\alpha_3 = 0.6$).

1. **Birth luminosity.** For each IMF, compute the total bolometric luminosity at birth,

$$L_{\rm SSP,0} = \int_{0.08\,M_\odot}^{150\,M_\odot} L_{\rm bol}(M)\, \xi(M)\, {\rm d}M, \tag{6}$$

   expressed in units of $L_\odot$. You may report either $L_{\rm SSP,0}$ for a specified $M_{\rm SSP}$ or the specific quantity $L_{\rm SSP,0}/M_{\rm SSP}$ in units of $L_\odot/M_\odot$.

2. **Late-time luminosity (only $\leq 1\,M_\odot$ stars survive).** Assume the population has aged to the point that all stars with $M > 1\,M_\odot$ have left the main sequence and contribute negligible bolometric luminosity. Approximate the late-time SSP luminosity as

$$L_{\rm SSP,\,1M_\odot} = \int_{0.08\,M_\odot}^{1\,M_\odot} L_{\rm bol}(M)\, \xi(M)\, {\rm d}M. \tag{7}$$

   Compute $L_{\rm SSP,\,1M_\odot}$ for each IMF (or $L_{\rm SSP,\,1M_\odot}/M_{\rm SSP}$).

3. **IMF comparison.** For each epoch, compute the luminosity ratio

$$\mathcal{R}_0 \equiv \frac{L_{\rm SSP,0}^{\rm (top\text{-}heavy)}}{L_{\rm SSP,0}^{\rm (canonical)}}, \qquad \mathcal{R}_{1M_\odot} \equiv \frac{L_{\rm SSP,\,1M_\odot}^{\rm (top\text{-}heavy)}}{L_{\rm SSP,\,1M_\odot}^{\rm (canonical)}}. \tag{8}$$

   Then compute, for each IMF separately, the fading factor

$$\mathcal{F} \equiv \frac{L_{\rm SSP,0}}{L_{\rm SSP,\,1M_\odot}}. \tag{9}$$

4. **Interpretation.** In one or two sentences, explain why a top-heavy IMF changes $L_{\rm SSP,0}$ much more than $L_{\rm SSP,\,1M_\odot}$.

OPTIONAL Problem 4: Core-collapse supernova rates

Use one of the following two IMFs. Use either the canonical IMF (IMF A) or the Salpeter IMF (IMF B) defined as:

$$\xi_A(m) = k\, m^{-2.35}\,, \qquad 0.1 \le m \le 100\,. \tag{10}$$

1. With the nomenclature for top/bottom-heavy/light IMFs from the glossary in Kroupa+26, how should the Salpeter IMF be classified?

1. Similarly to Problem 3, compute, for each IMF (A and B), the number of core-collapse supernovae per unit stellar mass formed,

$$\eta_{\rm SN} \;=\; \frac{\int_{10}^{100} \xi(m)\,{\rm d}m}{\int_{0.1}^{100} m\,\xi(m)\,{\rm d}m}\,, \tag{11}$$

   in units of number of supernovae per solar mass.

2. Convert $\eta_{\rm SN}$ into a present-day SN rate $R_{\rm SN}$ using a star formation rate of $\dot{M}_\star = 2\,M_\odot\,{\rm yr}^{-1}$. Report the answer in ${\rm SN\,century}^{-1}$.

3. Change only the upper SN progenitor limit from $m_{\rm SN,up} = 100$ to $m_{\rm SN,up} = 50$. Recompute the SN rates for both IMFs. Report the fractional change

$$\frac{R_{\rm SN}(50) - R_{\rm SN}(100)}{R_{\rm SN}(100)}\,. \tag{12}$$

4. Briefly compare your rates to an observed order-of-magnitude value of $R_{\rm SN} \approx 2$ to 3 per century for a Milky-Way-like disk. State one physical reason why the simple calculation might differ from observations.

OPTIONAL Problem 5: Oxygen from core-collapse supernovae

Assume each core-collapse supernova ejects a mass of newly produced oxygen $M_{\rm O}(m)$ that depends on the progenitor mass m as

$$M_{\rm O}(m) = \begin{cases} 2.4\,(m/25)^3, & 10 \le m \le 25, \\ 2.4\,(m/25)^2, & 25 \le m \le 100, \end{cases} \tag{13}$$

with $M_{\rm O}$ in units of $M_\odot$.

1. For each IMF (A and B), compute the IMF-weighted *average oxygen mass per supernova*,

$$\langle M_{\rm O}\rangle \;=\; \frac{\int_{10}^{100} M_{\rm O}(m)\,\xi(m)\,{\rm d}m}{\int_{10}^{100} \xi(m)\,{\rm d}m}\,. \tag{14}$$

2. Define the *oxygen yield per unit stellar mass formed* as

$$y_{\rm O} \;=\; \frac{\int_{10}^{100} M_{\rm O}(m)\,\xi(m)\,{\rm d}m}{\int_{0.1}^{100} m\,\xi(m)\,{\rm d}m}\,. \tag{15}$$

Compute $y_{\rm O}$ for each IMF (A and B). Explain in one sentence why $\dot{M}_\star$ does not appear in $y_{\rm O}$.

3. Repeat parts (1) and (2) with the modified assumption $m_{\rm SN,up} = 50$. Comment on whether changing $m_{\rm SN,up}$ impacts $R_{\rm SN}$ and $y_{\rm O}$ by similar factors.

4. Compare your $y_{\rm O}$ to the solar oxygen mass fraction $Z_\odot^{\rm O} = 0.006$. Do not build a full chemical evolution model. Just state whether $y_{\rm O}$ is smaller than, comparable to, or larger than $Z_\odot^{\rm O}$ for each IMF.

Conventions and useful integrals

Let m be the initial stellar mass in units of $M_\odot$. Let $\xi(m)$ be the IMF in units of "number of stars per unit mass".

For a constant star formation rate $\dot{M}_\star$ (in $M_\odot\,{\rm yr}^{-1}$), the Type II (core-collapse) supernova rate is

$$R_{\rm SN} \;=\; \dot{M}_\star\,\frac{\int_{m_{\rm SN,low}}^{m_{\rm SN,up}} \xi(m)\,{\rm d}m}{\int_{m_{\rm min}}^{m_{\rm max}} m\,\xi(m)\,{\rm d}m}\,. \tag{16}$$

The overall IMF normalization cancels in this ratio.

You may use

$$\int m^{-\alpha}\,{\rm d}m = \frac{m^{-\alpha}}{1-\alpha} \quad (\alpha \neq 1), \qquad \int m\,m^{-\alpha}\,{\rm d}m = \frac{m^{2-\alpha}}{2-\alpha} \quad (\alpha \neq 2). \tag{17}$$

Given numbers for this sheet

- Total stellar mass range: $m_{\rm min} = 0.1$, $m_{\rm max} = 100$.
- Fiducial star formation rate: $\dot{M}_\star = 5\,M_\odot\,{\rm yr}^{-1}$.
- Core-collapse progenitor range: $m_{\rm SN,low} = 10$ and first use $m_{\rm SN,up} = 100$.
- Conversion: 1 century = 100 yr.