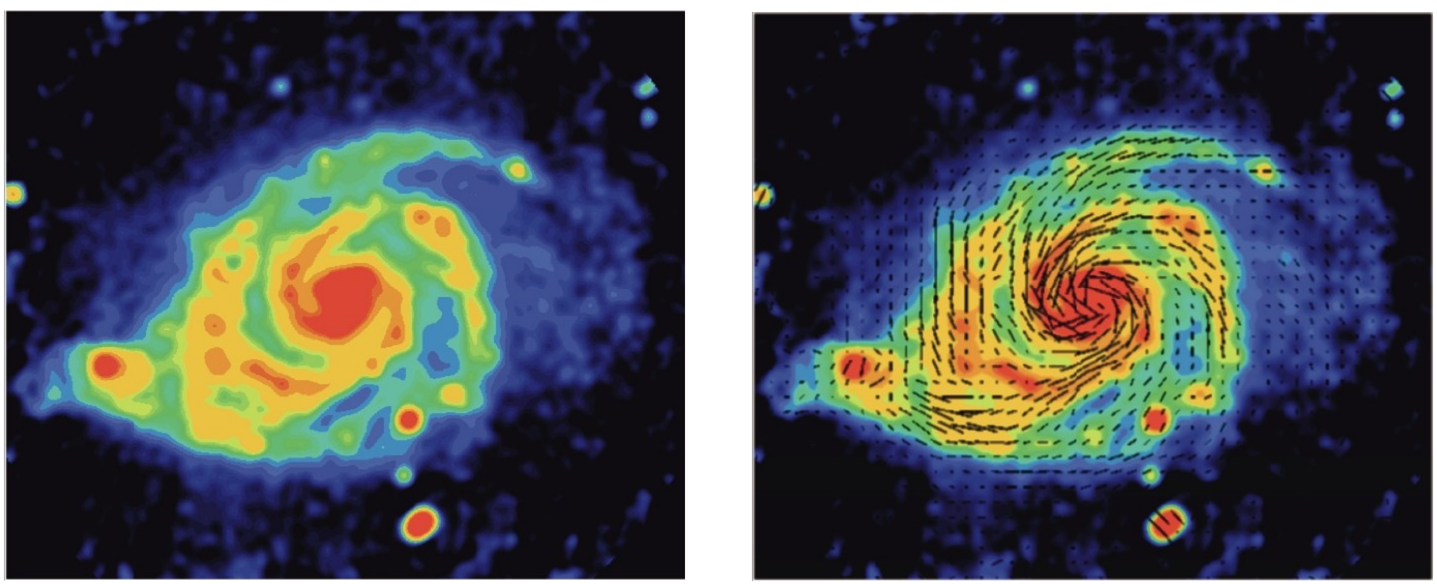
ASTRONOMICKÝ ÚSTAV
Akademie věd České republiky, v. v. i.

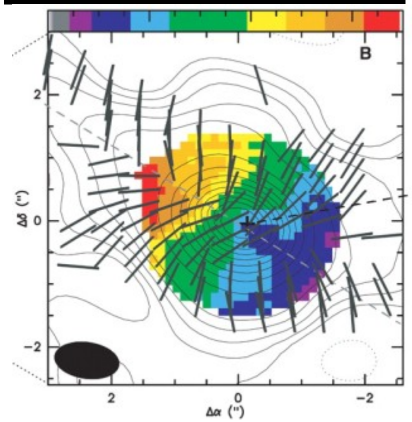
# Astrophysical polarimetry

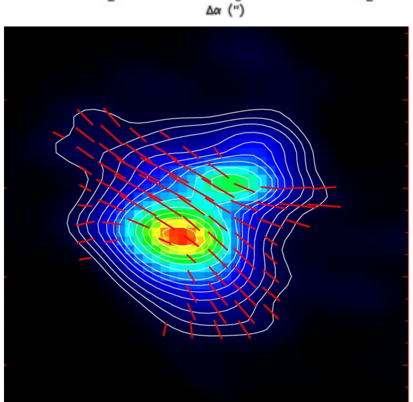
**Dr. Frédéric Marin**



Total radio continuum emission from the "Whirlpool" galaxy M51 (NRAO/AUI)

# Overview




- **I General introduction**
- **II Polarization : what is it and where can we observe it ?**
- **III Theory**
- **IV Polarization mechanisms**
- **V Observational techniques**
- **VI Modeling polarization**
- **VII Project about radio-loud quasars and polarization**

Any question ?
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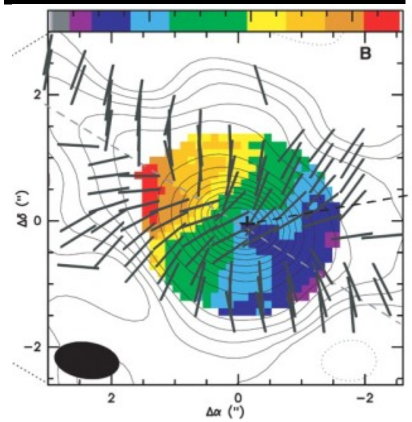


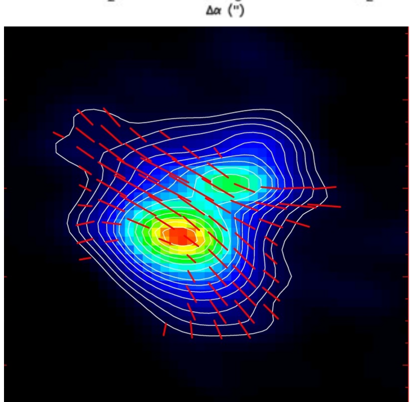


# Overview

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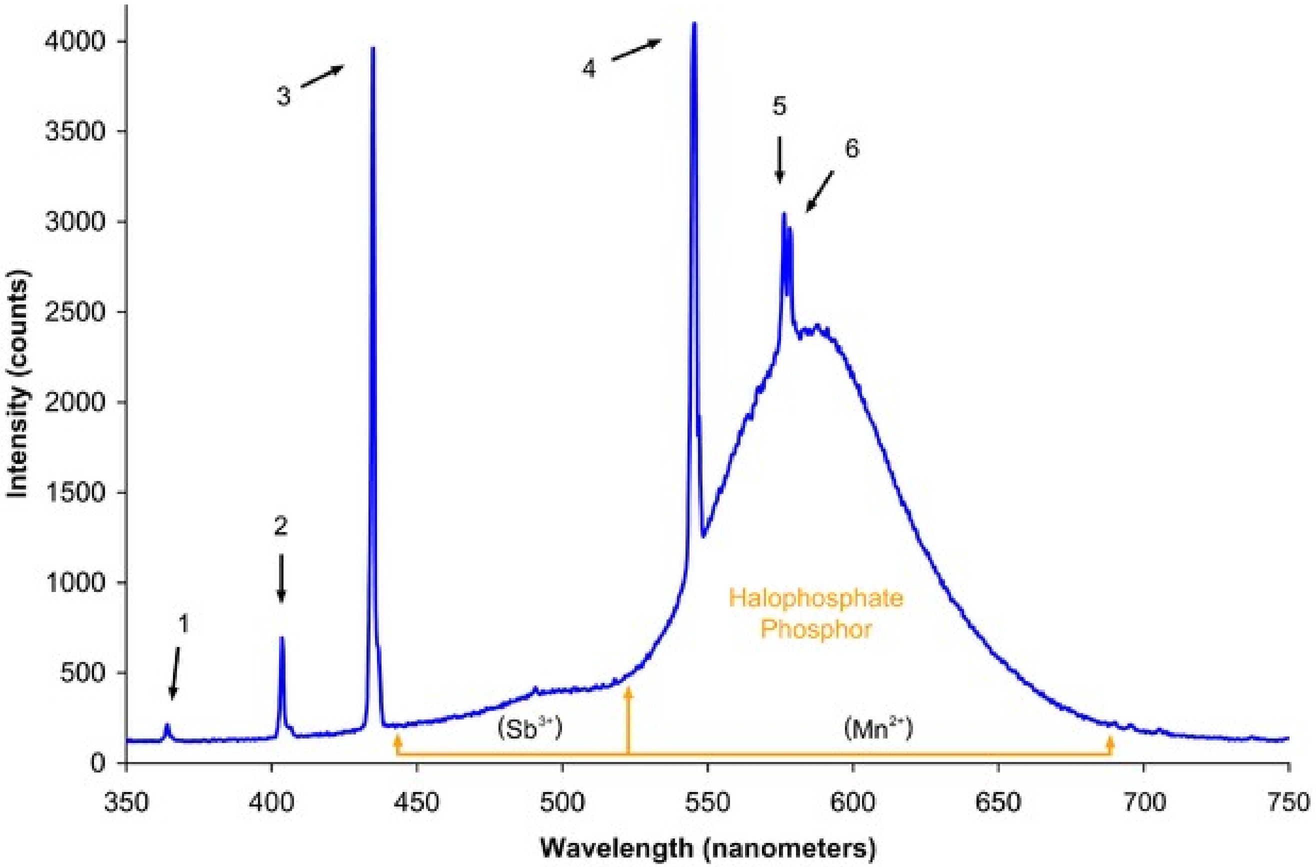
# Introduction

Astrophysics gets information from distant sources through light
→ intensity of the radiation as a function of wavelength/energy

*Fluorescent bulb*



# Introduction

Astrophysics gets information from distant sources through light
→ intensity of the radiation as a function of wavelength/energy

Intensity spectra (total flux) allow to probe:
- Distance (through line redshift)

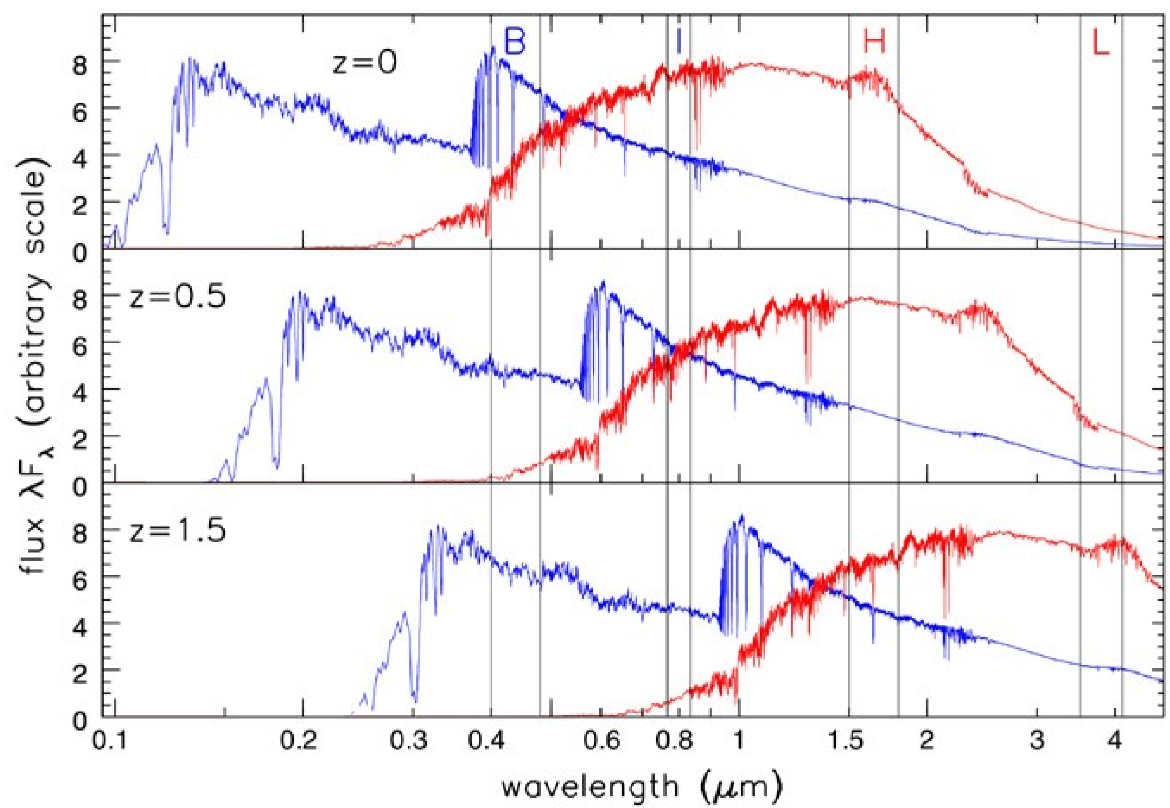
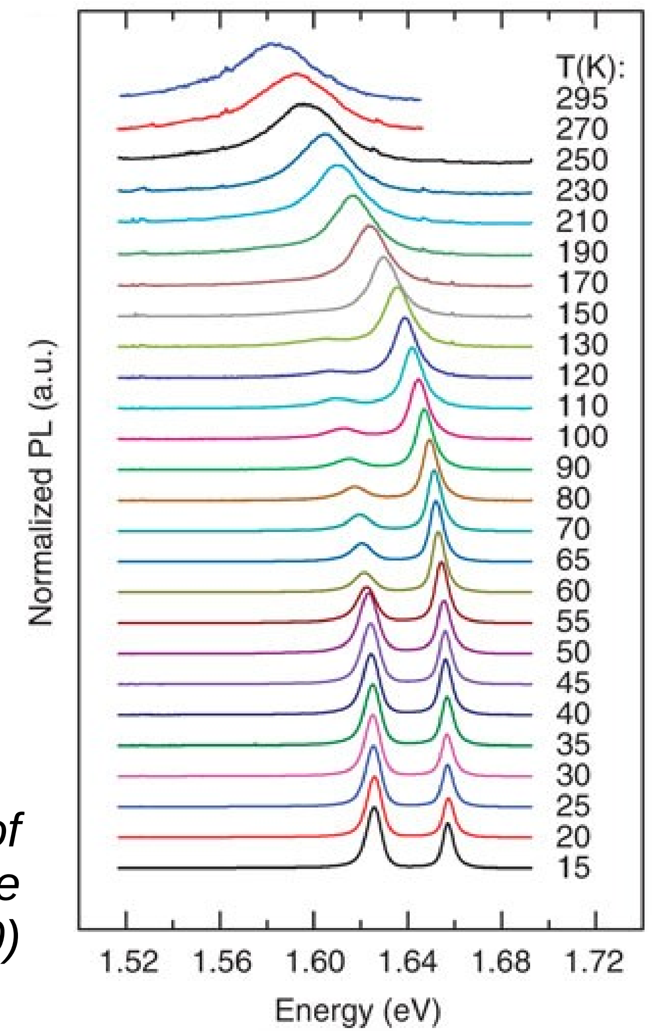



Fig 8.12 (S. Charlot) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

# Introduction

Astrophysics gets information from distant sources through light
→ intensity of the radiation as a function of wavelength/energy

Intensity spectra (total flux) allow to probe:
- Distance (through line redshift)
- Temperature




*Normalized photo-luminescence (PL) of monolayer MoSe2 versus temperature Ross et al. (2010)*

# Introduction

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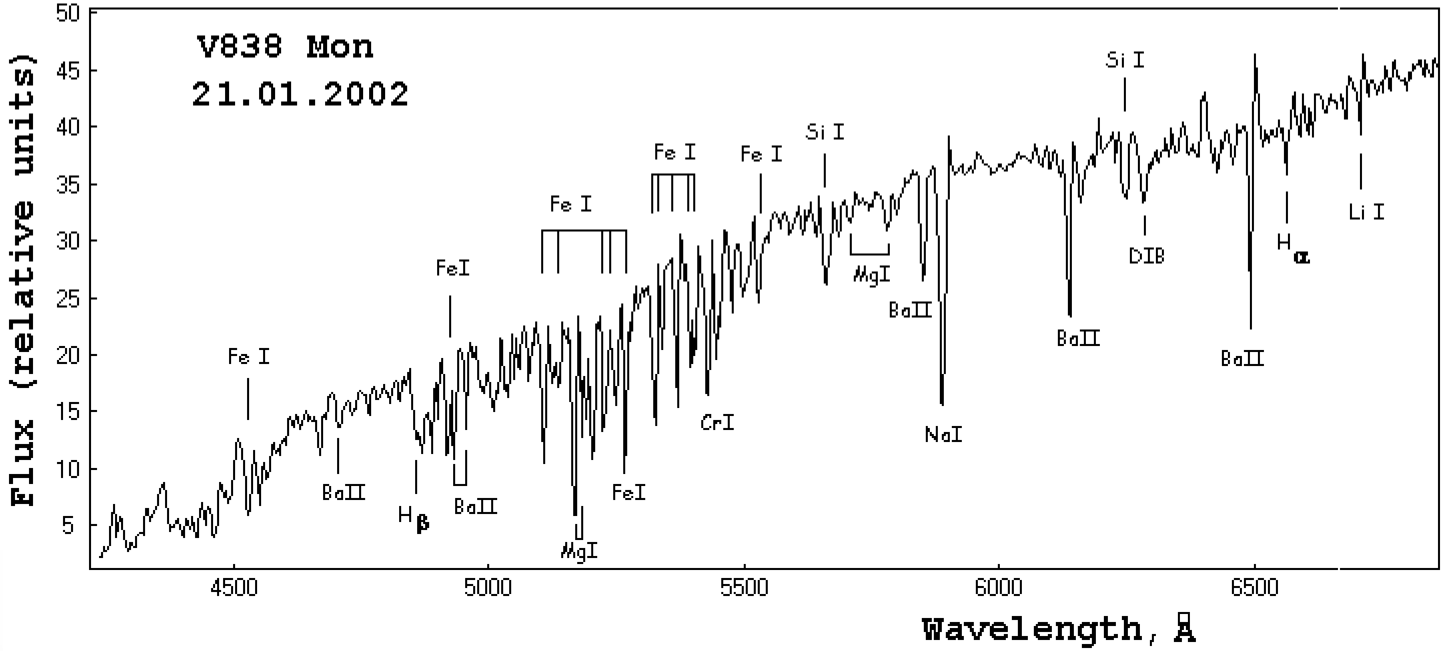
- Composition




*Spectrum of the nova V838 Mon in pre-maximum stage*
*Barsukova et al. (2007)*

# Introduction

Astrophysics gets information from distant sources through light
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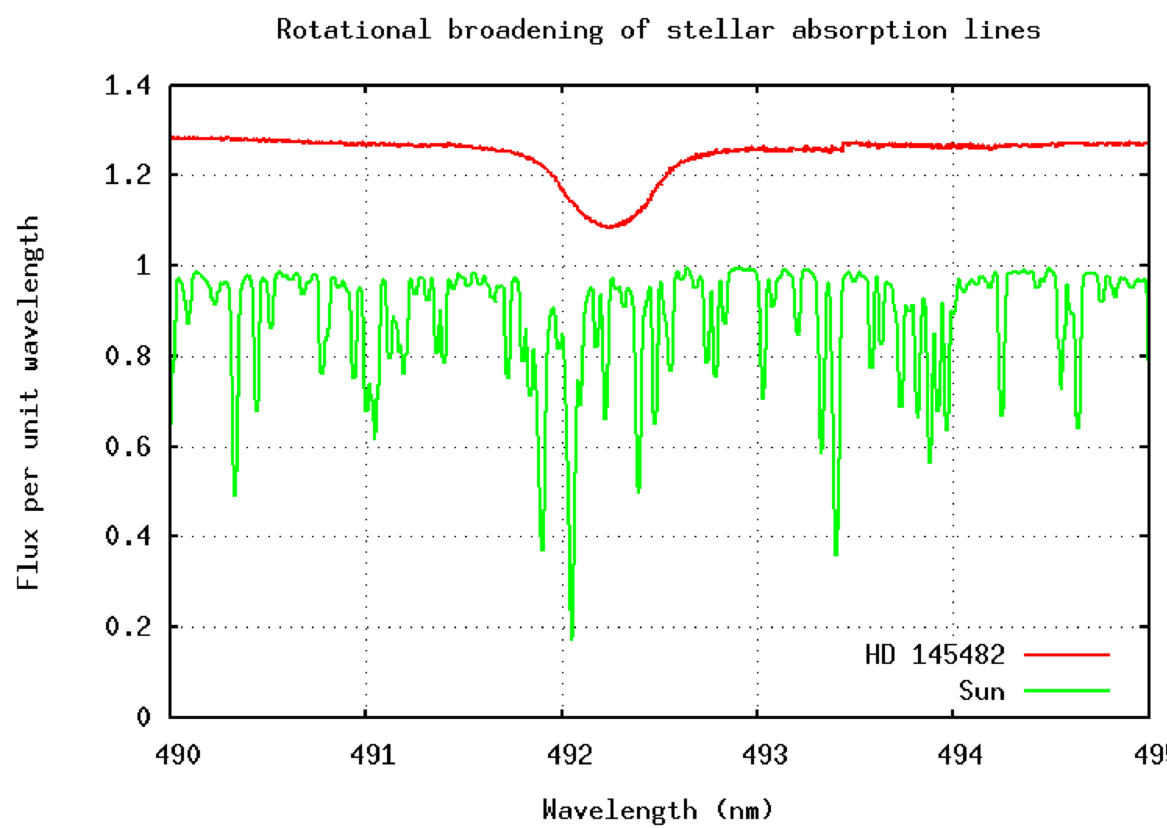
Intensity spectra (total flux) allow to probe:
- Distance (through line redshift)
- Temperature
- Composition
- Velocity of the gas



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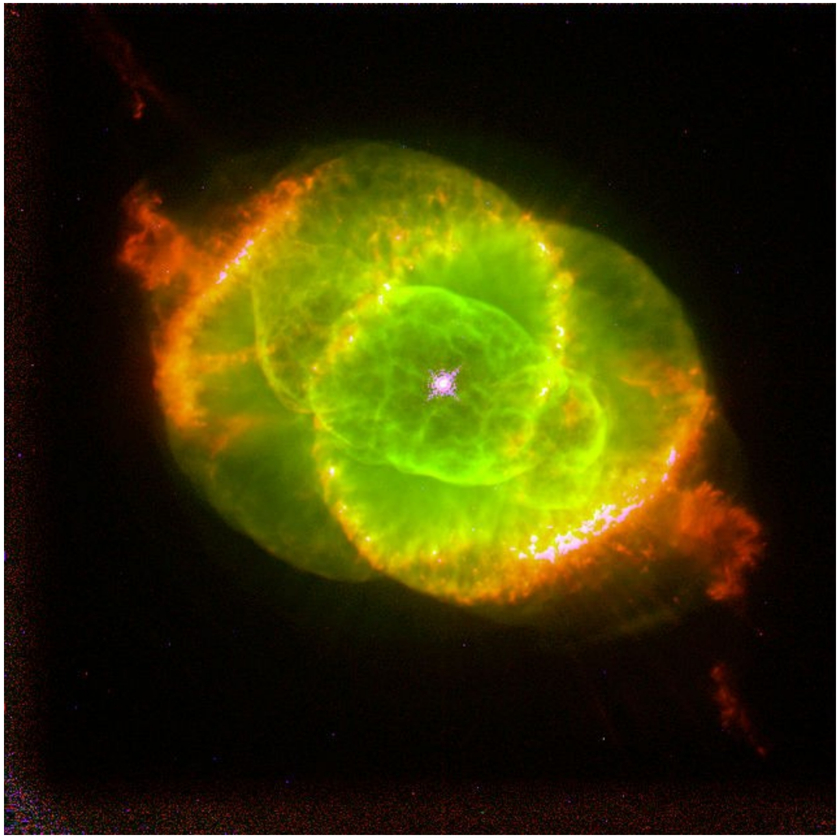
Cool maps and pictures



*Cat's Eye Nebula*
*HST (1994)*

# Introduction

Astrophysics gets information from distant sources through light

→ intensity of the radiation as a function of wavelength/energy

Intensity spectra (total flux) allow to probe:

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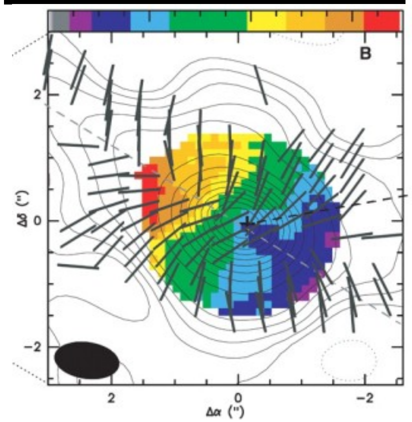
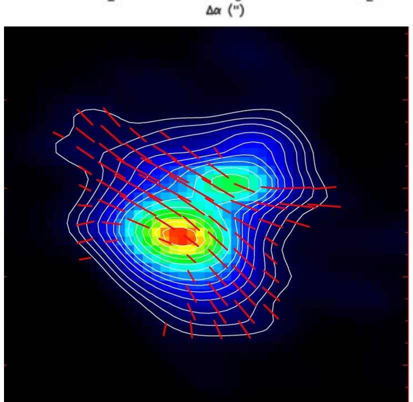
Cool maps and pictures

But deeply encoded in the electromagnetic radiation lies another essential information

POLARIZATION

# Overview

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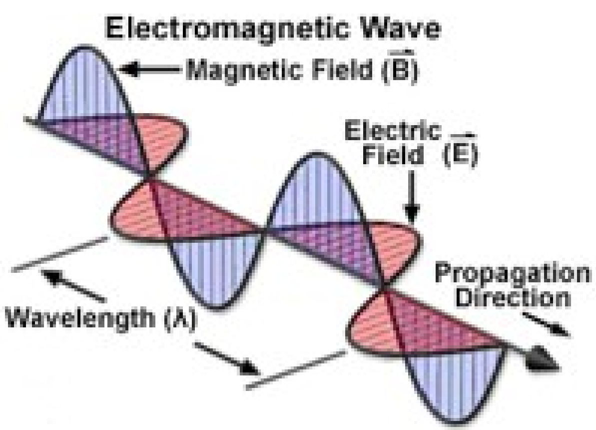
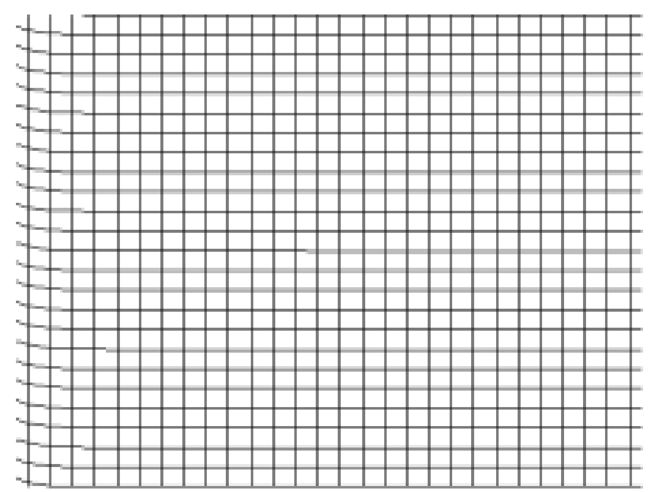
# Polarization

<u>The transverse nature of light</u>

Polarization is intrinsically connected with the transverse nature of light

The electric and magnetic field vectors oscillate (vibrate) perpendicularly (or right angled) to the direction of energy transfer

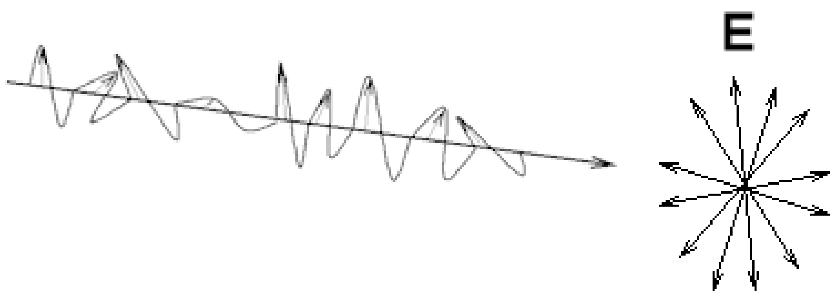


Examples: light, transverse seismic waves,
Waves in a guitar string

# Polarization

By definition, the electric (and the magnetic) vector oscillates randomly

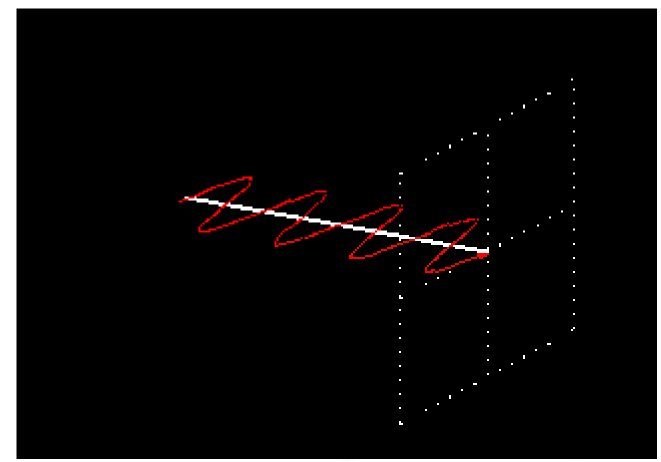


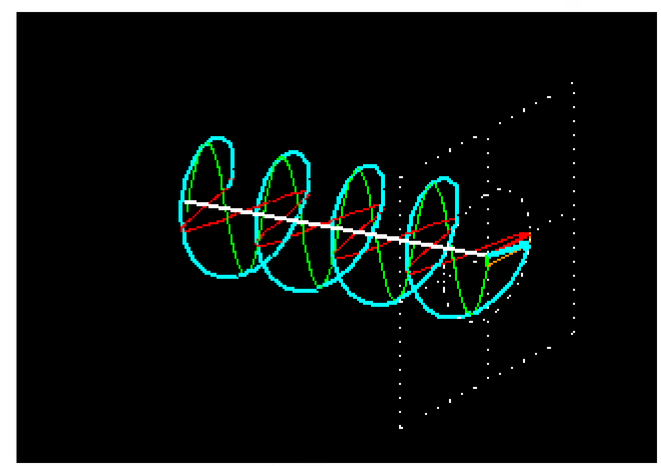

But nature is sometimes playful

Looking at the temporal evolution of tip of the electric vector

If the trend is found to be stationary

→ the wave is said to be polarized





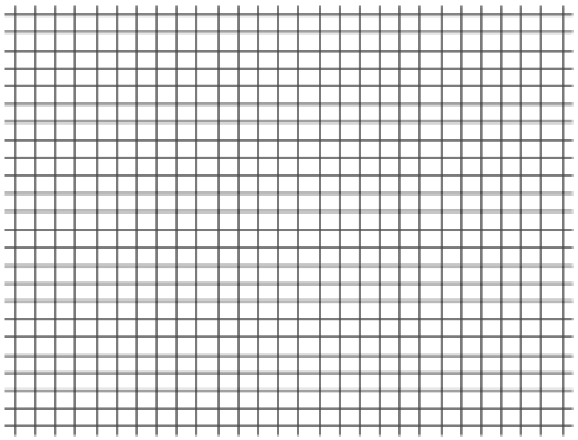
# Polarization

<u>Definition</u>

An electromagnetic wave is said to be polarized if the tip of its transverse (electric) vector is found not to vary during the measurement time

→ vectorial nature of light (Young 1801 and Fresnel 1821)

For this reason, polarization phenomena are inexistent for longitudinal waves (e.g. acoustic waves propagating in a gas/liquid)



# Polarization

Polarimetry

Polarization is thus a new information encoded in spatially asymmetric electromagnetic waves (along with intensity, frequency and phase)

But is it a useful information ?

If so, we should see it !

Well, while being very effective, the human vision is only sensitive to scalar quantities such as colors (modulation frequency / intensity)
→ we do not distinguish the vectorial nature

We need a polarimeter to detect/measure polarized light

# Polarization

<u>Polarimetry</u>

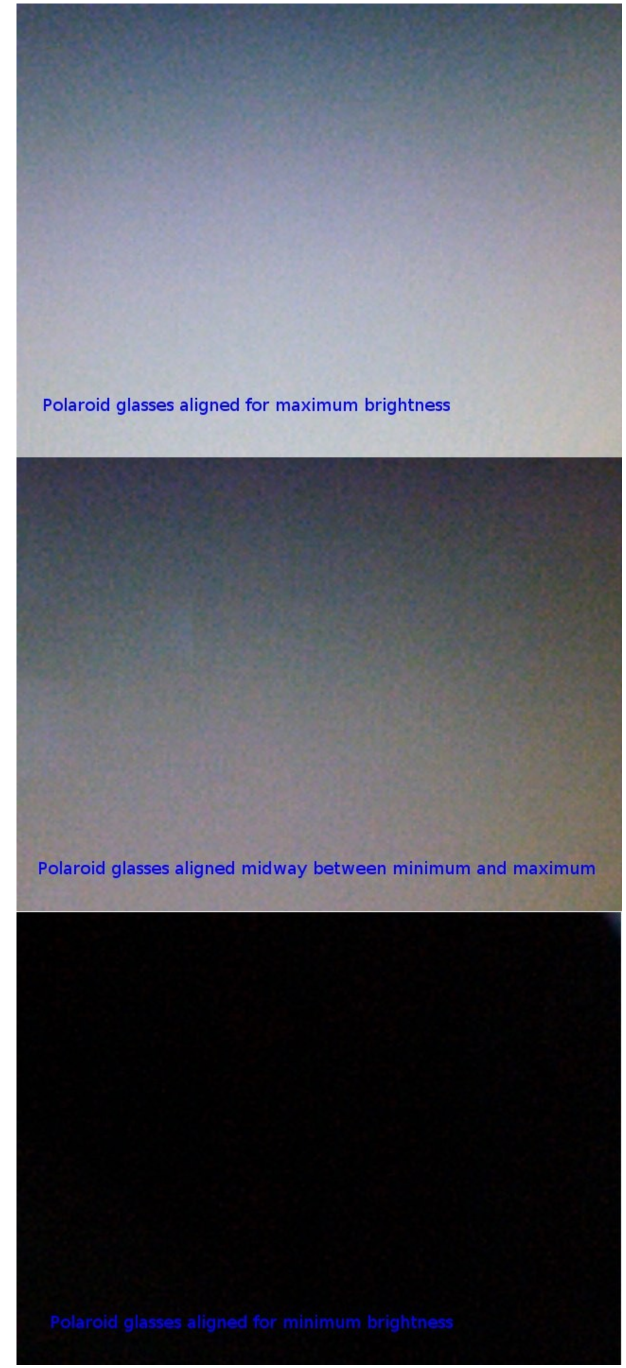
Polymer materials stretched in one direction (the polymer chains are aligned along one axis)

*Polarized light*

*Polaroid aligned with polarization*

*Polaroid aligned midway with pol.*

*Polaroid orthogonal to polarization*




Pol. dir. parallel to the chains
= strong absorption
Pol. dir. perpendicular
= weak absorption

# Polarization

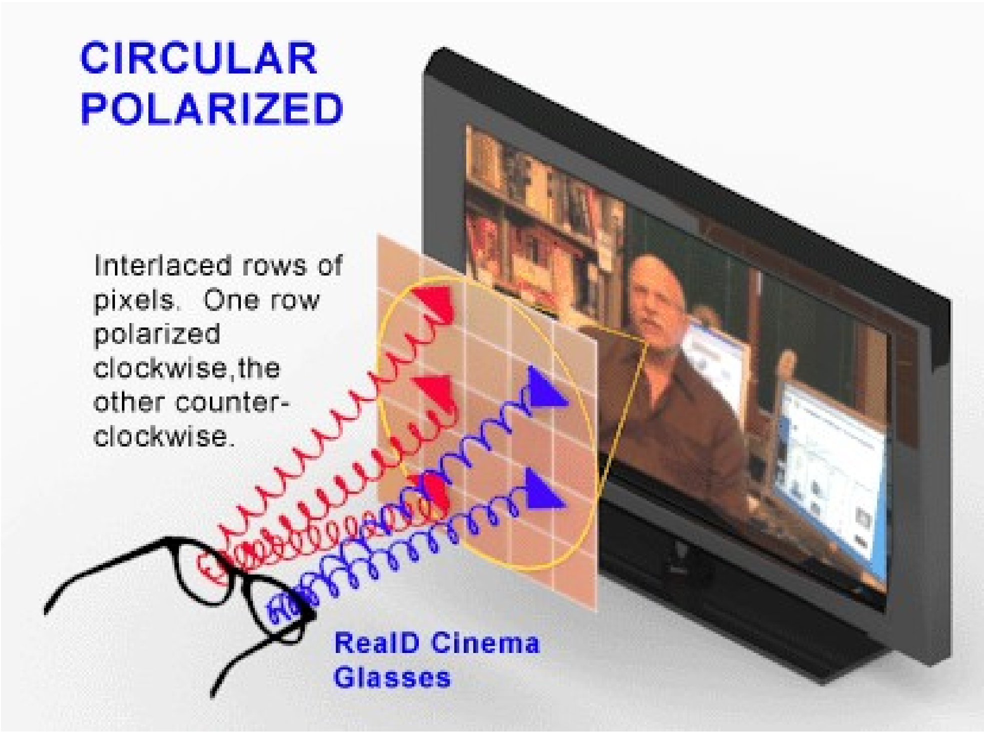
<u>A handful of examples</u>

Polarized 3D for the Cinema

The 3D movie projectors send polarized images to the screen, which reflect back to the viewer's polarized glasses

Polarized 3D glasses with two different lens polarizations, filtering the images to each eye respectively

However, because the lines are interlaced, the viewer sees half the resolution



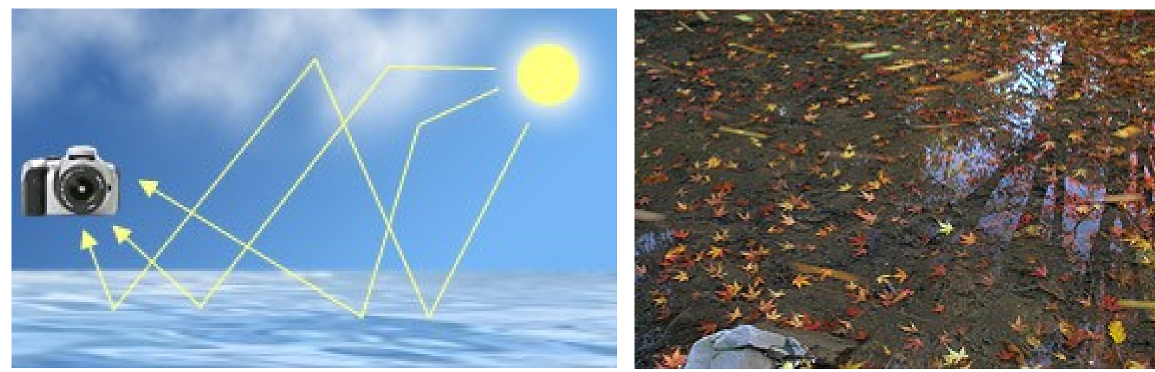
# Polarization

<u>A handful of examples</u>

Camera polarizing filters

*Without a polarizing filter*

Polarizers are placed in front of your camera lens, and work by filtering out sunlight which has been directly reflected toward the camera at specific angles



*With a polarizing filter*

polarizers increase color saturation
remove reflections
reduce image contrast

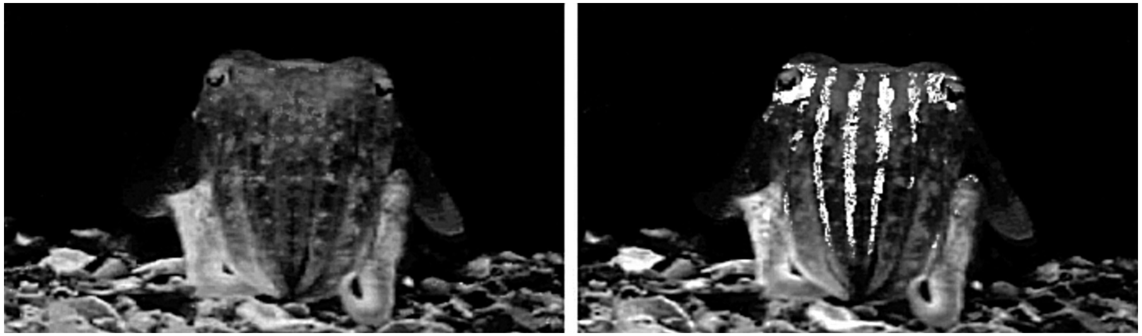
longer exposure time
need to be rotated
expensive

# Polarization

<u>A handful of examples</u>

Marine animal behavioral experiments

Cephalopods, squid, octopus and cuttlefish are known to be sensitive to the orientation of polarization of incoming light (orthogonal orientation of neighboring photoreceptors)



*A black and white intensity image and false color polarization image of a cuttlefish, Sepia officinalis Shashar et al. (1996a,b)*

Cuttlefish can display a prominent pattern of reflected polarized light, which alters predictably with behavioral context (aggression display, preying, copulation ...)

# Polarization

<u>Non related examples, but instructive or funny enough to be mentioned</u>

Economics: an economic process where middle-class jobs disappear more than those at the bottom and the top

Politics: the process by which the public opinion divides and goes to the extremes

Psychology: the process whereby a social or political group is divided into opposing sub-groups

Mathematics: polarization of an Abelian variety, of an algebraic form or polarization identity

Music: a progressive metal band from Los Angeles who released a CD called “Chasing the Light” (2012)

# Polarization

In astronomy and astrophysics

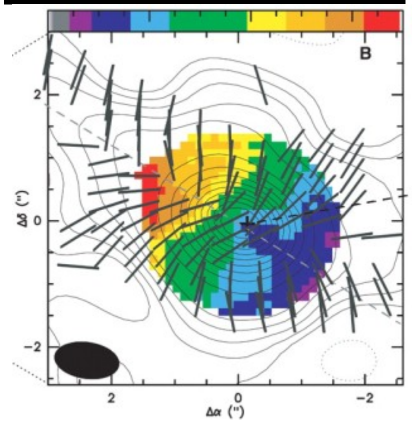
Polarization yields two more independent observables (P and $\psi$) than a pure intensity-spectrum

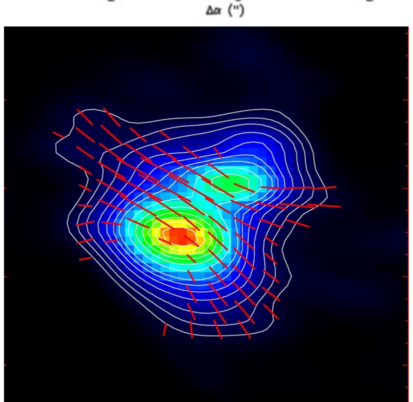
As said before, polarization is produced by asymmetry (either intrinsic to the source or along the observer line of sight)

Geometric asymmetries or the presence of magnetic fields are the most common sources of asymmetries in the distribution of (scattered) radiation

As most of the celestial radiation sources are either magnetized or asymmetrical, polarimetry can be used as a diagnostic tool from (proto-)stars to (extra-)solar planets, ISM*/IGM** to astrobiology

*Interstellar medium
**Intergalactic medium

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# Theory

<u>The wave equation</u>

The concept of light as a wave (in particular a transverse wave) is fundamental to the phenomena of polarization and propagation

Consider a homogeneous string (length *l*) fixed at both ends and under a tension $T_0$. The lateral displacements are assumed to be small compared with *l*.

$\theta$ is a small angle ($\theta \sim \tan\theta$) between any small segment of the string and the straight line joining the point of support.

x,y plane

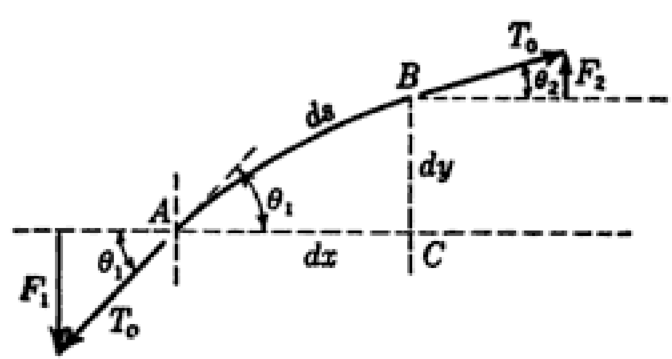



**Figure 2-1** Derivation of the wave equation. Motion of a string under tension.

# Theory

The wave equation

Differential equation of motion obtained by considering small element *ds* of the string (AB segment)

y-component of the force acting on *ds* is characterized by $F_1$ and $F_2$

Since $\theta_1$ and $\theta_2$ are small, then:

*Partial derivatives since* y *depends on time* t *and on distance* x

$$F_1 = T_0 \sin\theta_1 \simeq T_0 \tan\theta_1 = T_0\left(\frac{\partial y}{\partial x}\right)_A$$

$$F_2 = T_0 \sin\theta_2 \simeq T_0 \tan\theta_2 = T_0\left(\frac{\partial y}{\partial x}\right)_B$$

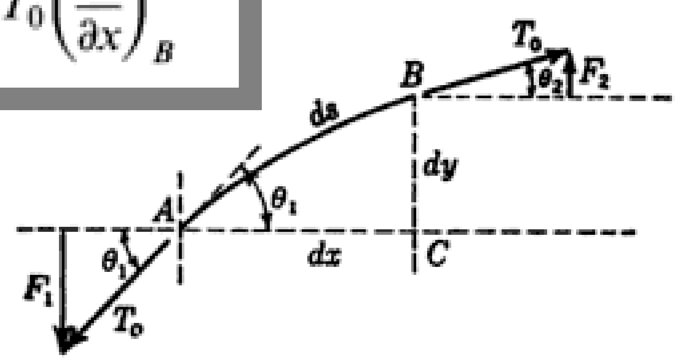



**Figure 2-1** Derivation of the wave equation. Motion of a string under tension.

# Theory

The wave equation

Using Taylor's expansion theorem:

$$\left(\frac{\partial y}{\partial x}\right)_A = \frac{\partial y}{\partial x} - \left[\frac{\partial}{\partial x}\frac{\partial y}{\partial x}\right]\frac{dx}{2} = \frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial x^2}\frac{dx}{2}$$

$$\left(\frac{\partial y}{\partial x}\right)_B = \frac{\partial y}{\partial x} + \left[\frac{\partial}{\partial x}\frac{\partial y}{\partial x}\right]\frac{dx}{2} = \frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2}\frac{dx}{2}$$

In which the derivatives without subscripts are evaluated at the midpoint of *ds*

The resultant force in the y-direction is:

$$F_2 - F_1 = T_0\left(\frac{\partial^2 y}{\partial x^2}\right)dx$$

# Theory

The wave equation

Now consider ρ (mass per unit length of the string). The inertial reaction of the element *ds* is: ρ*ds*($\partial^2$y/$\partial^2$t)

For small displacements: *ds* ~ *dx*

Then:

$$\frac{\partial^2 y}{\partial t^2} = \frac{T_0}{\rho}\frac{\partial^2 y}{\partial x^2}$$

→ Wave equation in 1D

In optics, y(x,t) is equated with the optical disturbance u(x,t)
Plus, one can prove that:

$$v^2 = \frac{T_0}{\rho}$$

# Theory

<u>The wave equation</u>

Let's rewrite

$$\frac{\partial^2 y}{\partial t^2} = \frac{T_0}{\rho}\frac{\partial^2 y}{\partial x^2} \longleftrightarrow \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2}$$

→ describes the 1D propagation of an optical disturbance u(x,t) in a direction *x* at a time *t*

Generalized to 3D:

$$\frac{\partial^2 u(r,t)}{\partial x^2} + \frac{\partial^2 u(r,t)}{\partial y^2} + \frac{\partial^2 u(r,t)}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 u(r,t)}{\partial t^2}$$

with $r = (x^2+y^2+z^2)^{1/2}$

# Theory

<u>The wave equation</u>

In a simpler form, it becomes the 3D wave equation:

$$\nabla^2 u(r,t) = \frac{1}{v^2}\frac{\partial^2 u(r,t)}{\partial t^2}$$

With the Laplacian operator:

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

# Theory

<u>The polarization ellipse</u>

Following mechanics and equating an optical to an isotropic elastic medium, 3 independent oscillations should exist ($u_x(r,t)$, $u_y(r,t)$ and $u_z(r,t)$)

Then:

$$\nabla^2 u_i(r,t) = \frac{1}{v^2}\frac{\partial^2 u_i(r,t)}{\partial t^2} \qquad i = x, y, z$$

Where *v* is the velocity propagation of the oscillation and **r** = **r**(x,y,z)

Naturally:

$$u_x(\mathbf{r},t) = u_{0x}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_x)$$
$$u_y(\mathbf{r},t) = u_{0y}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_y)$$
$$u_z(\mathbf{r},t) = u_{0z}\cos(\omega t - \mathbf{k}\cdot\mathbf{r} + \delta_z)$$

Phase (arbitrary)

Angular frequency

Maximum amplitude

# Theory

<u>The polarization ellipse</u>

In a Cartesian system, $u_x(\mathbf{r},t)$ and $u_y(\mathbf{r},t)$ are said to be transverse and $u_z(\mathbf{r},t)$ longitudinal (propagation in the z-direction)

Light is a transverse wave !

$$u_x(\mathbf{r}, t) = u_{0x} \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_x)$$

$$u_y(\mathbf{r}, t) = u_{0y} \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_y)$$

~~$$u_z(\mathbf{r}, t) = u_{0z} \cos(\omega t - \mathbf{k} \cdot \mathbf{r} + \delta_z)$$~~

Rewriting:

$$E_x(z, t) = E_{0x} \cos(\tau + \delta_x)$$

$$E_y(z, t) = E_{0y} \cos(\tau + \delta_y)$$

with $\tau = \omega t - kz$ (propagator)

# Theory

The polarization ellipse

$$\frac{E_x}{E_{0x}} = \cos\tau\cos\delta_x - \sin\tau\sin\delta_x$$

$$\frac{E_y}{E_{0y}} = \cos\tau\cos\delta_y - \sin\tau\sin\delta_y$$

$$\frac{E_x}{E_{0x}}\sin\delta_y - \frac{E_y}{E_{0y}}\sin\delta_x = \cos\tau\sin(\delta_y - \delta_x)$$

$$\frac{E_x}{E_{0x}}\cos\delta_y - \frac{E_y}{E_{0y}}\cos\delta_x = \sin\tau\sin(\delta_y - \delta_x)$$

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\delta = \sin^2\delta$$

(with $\delta = \delta_y - \delta_x$)

# Theory

<u>The polarization ellipse</u>

$$\frac{E_x}{E_{0x}} = \cos\tau\cos\delta_x - \sin\tau\sin\delta_x$$

$$\frac{E_y}{E_{0y}} = \cos\tau\cos\delta_y - \sin\tau\sin\delta_y$$

$$\frac{E_x}{E_{0x}}\sin\delta_y - \frac{E_y}{E_{0y}}\sin\delta_x = \cos\tau\sin(\delta_y - \delta_x)$$

$$\frac{E_x}{E_{0x}}\cos\delta_y - \frac{E_y}{E_{0y}}\cos\delta_x = \sin\tau\sin(\delta_y - \delta_x)$$

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\delta = \sin^2\delta$$

Equation of the polarization ellipse (with $\delta = \delta_y - \delta_x$)

# Theory

The polarization ellipse

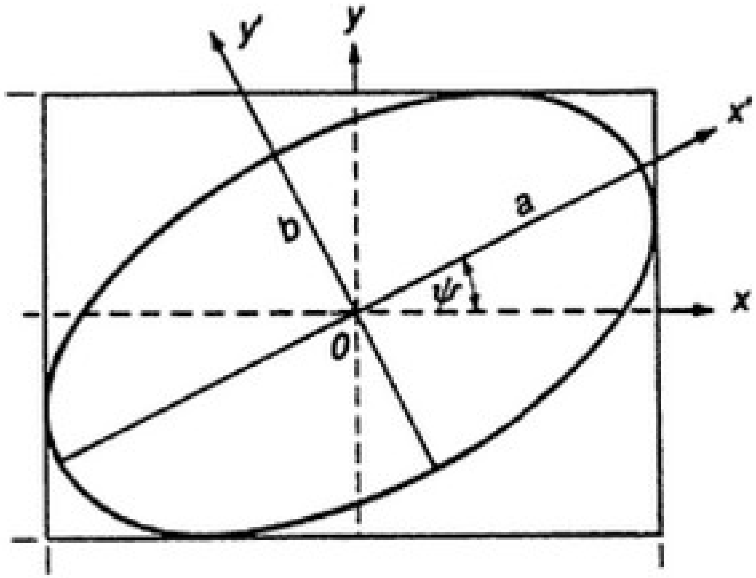



**Figure 3-3** The rotated polarization ellipse.

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - 2\frac{E_x}{E_{0x}}\frac{E_y}{E_{0y}}\cos\delta = \sin^2\delta$$

Rotated ellipse
($\Psi$ = angle of rotation)

$$E'_x = E_x \cos\psi + E_y \sin\psi$$

$$E'_y = -E_x \sin\psi + E_y \cos\psi$$

$$E'_x = a\cos(\tau + \delta')$$

$$E'_y = \pm b\sin(\tau + \delta')$$

$$\frac{E_x'^2}{a^2} + \frac{E_y'^2}{b^2} = 1$$

# Theory

The polarization ellipse

We replace the previous terms inside

$$\frac{E_x}{E_{0x}} = \cos(\tau + \delta_x)$$
$$\frac{E_y}{E_{0y}} = \cos(\tau + \delta_y)$$

And after several steps, equating propagator coefficients, we find:

$$\pm ab = E_{0x} E_{0y} \sin \delta$$

$$a^2 + b^2 = E_{0x}^2 + E_{0y}^2$$

$$(E_{0x}^2 - E_{0y}^2) \sin 2\psi = 2E_{0x} E_{0y} \cos \delta \cos 2\psi$$

Leading to:

$$\tan 2\psi = \frac{2E_{0x} E_{0y} \cos \delta}{E_{0x}^2 - E_{0y}^2}$$

# Theory

<u>The polarization ellipse</u>

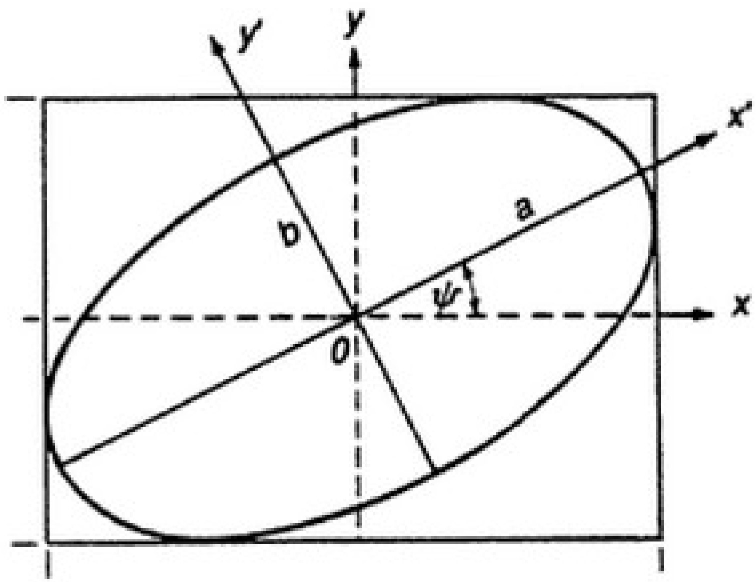



**Figure 3-3** The rotated polarization ellipse.

Introducing the auxiliary angle α ($0 \leq \alpha \leq \pi/2$) such as:

$$\tan\alpha = \frac{E_{0y}}{E_{0x}}$$

So we can re-write:

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 E_{0y}^2}\cos\delta = \frac{2\tan\alpha}{1-\tan^2\alpha}\cos\delta$$

Leading to:

$$\tan 2\psi = (\tan 2\alpha)\cos\delta$$

(if $\delta = 0$ or $\pi \rightarrow \Psi = \pm\,\alpha$)

(if $\delta = \pi/2$ or $3\pi/2 \rightarrow \Psi = 0$)

# Theory

The polarization ellipse

Finally introducing the ellipticity angle such as: $\tan\chi = \frac{\pm b}{a} \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}$

Leads to: $$\frac{\pm 2ab}{a^2+b^2} = \frac{2E_{0x}E_{0y}}{E_{0x}^2+E_{0y}^2}\sin\delta = (\sin 2\alpha)\sin\delta$$

And: $$\sin 2\chi = (\sin 2\alpha)\sin\delta$$

If b = 0 → $\chi$ is null
→linear polarization

If b = a → $\chi$ is maximum
→circular polarization

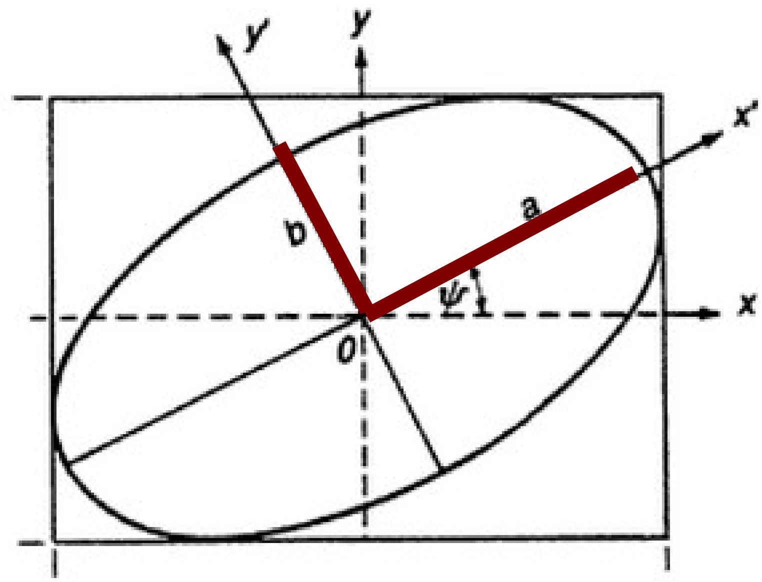



**Figure 3-3** The rotated polarization ellipse.

# Theory

The polarization ellipse

Summarizing the important equations:

| | | | |
|---|---|---|---|
| Polarization angle | ⟵ | $\tan 2\psi = (\tan 2\alpha)\cos\delta$ | $0 \le \psi \le \pi$ |
| Polarization ellipticity | ⟵ | $\sin 2\chi = (\sin 2\alpha)\sin\delta$ | $-\frac{\pi}{4} < \chi \le \frac{\pi}{4}$ |
| | | where $0 \le \alpha \le \pi/2$ and | |
| Amplitude of polarization | ⟵ | $a^2 + b^2 = E_{0x}^2 + E_{0y}^2$ | |
| Auxiliary polarization angle | ⟵ | $\tan\alpha = \frac{E_{0y}}{E_{0x}}$ | |
| | | $\tan\chi = \frac{\pm b}{a}$ | |

# Theory

Stokes formalism

Consider a pair of monochromatic plane waves, orthogonal to each other at a point in space (z = 0). They are represented by the (now known) equation:

$$\frac{E_x^2(t)}{E_{0x}^2} + \frac{E_y^2(t)}{E_{0y}^2} - \frac{2E_x(t)E_y(t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta$$

However, the amplitudes $E_x(t)$ and $E_y(t)$ are time-dependent. To measure them, we need to <average> them over the observational time:

$$\frac{\langle E_x^2(t)\rangle}{E_{0x}^2} + \frac{\langle E_y^2(t)\rangle}{E_{0y}^2} - \frac{2\langle E_x(t)E_y(t)\rangle}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta$$

# Theory

Stokes formalism

We use:

$$\langle E_i(t)E_j(t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(t)E_j(t)\,dt \qquad i,j=x,y$$

And a different form of the previous equation (multiplied by $4E^2_{0x}E^2_{0y}$)

$$4E_{0y}^2\langle E_x^2(t)\rangle + 4E_{0x}^2\langle E_y^2(t)\rangle - 8E_{0x}E_{0y}\langle E_x(t)E_y(t)\rangle\cos\delta = (2E_{0x}E_{0y}\sin\delta)^2$$

We find:

$$\langle E_x^2(t)\rangle = \frac{1}{2}E_{0x}^2$$

$$\langle E_y^2(t)\rangle = \frac{1}{2}E_{0y}^2$$

$$\langle E_x(t)E_y(t)\rangle = \frac{1}{2}E_{0x}E_{0y}\cos\delta$$

# Theory

Stokes formalism

Substituting: $$2E_{0x}^2E_{0y}^2 + 2E_{0x}^2E_{0y}^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2$$

And expressing the result in terms of intensity $E_0$:

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2$$

# Theory

Stokes formalism

Substituting:

$$2E_{0x}^2E_{0y}^2 + 2E_{0x}^2E_{0y}^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2$$

And expressing the result in terms of intensity $E_0$:

$$(E_{0x}^2 + E_{0y}^2)^2 - (E_{0x}^2 - E_{0y}^2)^2 - (2E_{0x}E_{0y}\cos\delta)^2 = (2E_{0x}E_{0y}\sin\delta)^2$$

$S_0$ $S_1$ $S_2$ $S_3$

**Stokes polarization parameters**

$$S_0^2 = S_1^2 + S_2^2 + S_3^2$$

# Theory

*George Gabriel Stokes (1819 – 1903)*

<u>Stokes formalism</u>

$S_0$,$S_1$,$S_2$ and $S_3$ are related to intensities
→ measurable quantities

$S_0$ = total intensity of the photon flux
$S_1$ = difference between the vertical and horizontal polarization state
$S_2$ = difference between the linear polarization oriented at +45° and -45° from the vertical polarization state
$S_3$ = difference between the clockwise and anti-clockwise rotational directions

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

*Stokes vector*

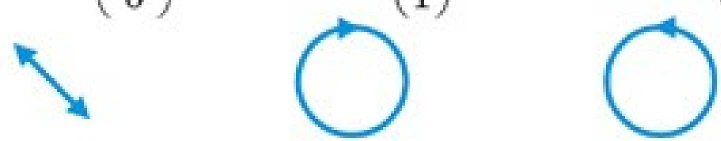
# Theory

<u>Stokes formalism</u>

The Stokes vectors for the degenerate polarization states are readily found using the previous definitions and equations:

$$S_{\mathrm{LHP}} = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad S_{\mathrm{LVP}} = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \quad S_{\mathrm{L+45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$S_{\mathrm{L-45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \quad S_{\mathrm{RCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad S_{\mathrm{LCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$



# Theory

Stokes formalism

The Stokes parameters can be shown to be related to the ellipse's orientation and ellipticity angles ψ and χ, respectively:

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right), \qquad 0 \le \psi \le \pi,$$

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right), \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}.$$

And, finally, the polarization degree of a monochromatic wave can be evaluated with:

$$P = \frac{I_{\mathrm{pol}}}{I_{\mathrm{tot}}} = \frac{(S_1^2 + S_2^2 + S_3^2)^{1/2}}{S_0} \qquad 0 \le P \le 1$$

# Theory

<u>Stokes formalism</u>

The Stokes parameters can be shown to be related to the ellipse's orientation and ellipticity angles ψ and χ, respectively:

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right), \qquad 0 \le \psi \le \pi,$$

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right), \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}.$$

And, finally, the polarization degree of a <u>quasi-monochromatic wave</u> (a pack of photons) can be evaluated with:

$$S_0^2 \ge S_1^2 + S_2^2 + S_3^2$$

# Theory

Stokes formalism

It follows that any wave characterized by a Stokes vector can be decomposed into a completely depolarized and a completely polarized counterpart:

$$\begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} S_0 - \sqrt{S_1^2 + S_2^2 + S_3^2} \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sqrt{S_1^2 + S_2^2 + S_3^2} \\ S_1 \\ S_2 \\ S_3 \end{bmatrix}$$

# Theory

Mueller matrices

The Stokes parameters are used to fully characterize the state of polarization of a light wave. Mueller (1948), meanwhile, showed that there was a linear relationship between the in-coming ($S_{in}$) and out-going ($S_{out}$) Stokes vector during a scattering event:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}^{out} = \underbrace{\begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}}_{\text{Mueller matrix}} \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}^{in}$$

Mueller matrix

# Theory

Mueller matrices

The Mueller matrix [M], whose elements $m_{ij}$ (i, j = 0, 1, 2, 3) are real, is directly connected to the measurement. It may be regarded as the polarization transfer matrix of a given environment. Knowing [M], the polarization state of the system output can be predicted when the incident conditions are known

The matrix properties determine the Mueller matrix of an optical system formed by a succession of elements. It is the product of the matrices of the elements constituting the system.

Considering n optical elements traversed by a light ray in the order 1, 2; ... n, then the resulting Mueller matrix is:

$$[M] = [M_n][M_{n-1}][M_{n-2}]...[M_2][M_1]$$

# Theory

Mueller matrices: a simple example

In order to determine the correct Stokes vector for a photon scattered off an electron (Thomson scattering approximation), we rotate the incident vector into the frame of reference of the moving photon using a Mueller matrix, L(ψ), apply the scattering matrix R(Θ), where Θ is the angle between the incident and scattered radiation (α = cos Θ), and then rotate the vector back into the original frame of reference:

$$\mathbf{S} = \mathbf{L}(\pi - i_2)\mathbf{R}\mathbf{L}(-i_1)\mathbf{S}'$$

$$\mathbf{L}(\psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\psi & \sin 2\psi & 0 \\ 0 & -\sin 2\psi & \cos 2\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}(\Theta) = \frac{3}{4}\begin{bmatrix} \alpha^2+1 & \alpha^2-1 & 0 & 0 \\ \alpha^2-1 & \alpha^2+1 & 0 & 0 \\ 0 & 0 & 2\alpha & 0 \\ 0 & 0 & 0 & 2\alpha \end{bmatrix}.$$