WINDS OF HOT MASSIVE STARS

II Lecture: Basic theory of winds of hot massive stars

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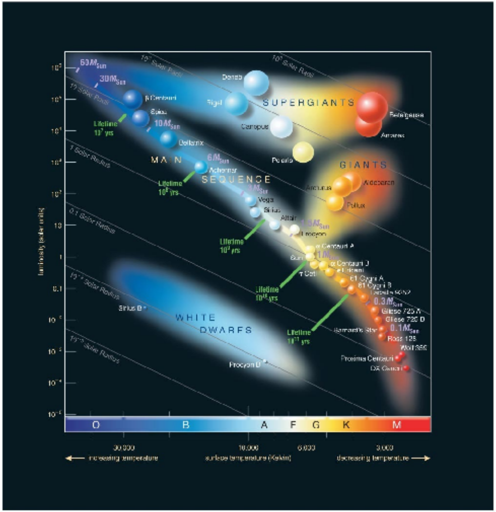
Selected Topics in Astrophysics
Faculty of Mathematics and Physics
October 16, 2013
Prague

Properties of winds of hot massive stars

- EXTREMELY LUMINOUS
  spectral types A, B, and O;
  $L \gtrsim 10^2\ [L_\odot]$
  W-R, LBV, B[e] stars
- HOT - $T_{\rm eff} \gtrsim 8\,000$ [K]
- MASSIVE - $M \gtrsim 2\ [M_\odot]$
- SHORT LIFETIMES
  ($\sim 10^6$ yr)
- END IN SUPERNOVA EXPLOSION
- HAVE WIND

H-R diagram



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Typical parameters for O-type stars and their winds

Parameter	Sun	O-type stars
L [$L_\odot$]	1	$\sim 10^6$
T_{eff}[K]	6000	$\gtrsim 30\,000$
M [$M_\odot$]	1	$\gtrsim 8$
total life time [yr]	10^{10}	$\sim 10^7$
T_{wind}[K]	10^6	$\sim 10^4$
$\dot{M}[M_\odot\ yr^{-1}]$	10^{-14}	$\sim 10^{-6}$
v_∞[km s^{-1}]	400 (700)	$\sim 10^2 - 10^3$

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- TYPICAL v_∞ - from 200 km s^{-1}(for A-supergiant) to 3 000 km s^{-1}(for early O-stars)

Properties of winds of hot massive stars

- Hot stars emit their peak radiation in the UV wavelength region
  Wien's displacement law

$$\lambda_{\mathrm{max}} T = b$$

$$b = 0.29\,\mathrm{cm\,K}; \quad T = 30\,000\,\mathrm{K} \;\Rightarrow\; \lambda_{\mathrm{max}} = 960\,\text{Å}$$

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- The outer atmospheres of hot stars have plenty of absorption lines in the ultraviolet, e.g., resonance lines from N V $\lambda\lambda$ 1239, 1243 Å, Si IV $\lambda\lambda$ 1394, 1403 Å, C IV $\lambda\lambda$ 1548, 1551 Å(see Morton, 1967)

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- Massive hot stars are luminous $\Rightarrow$ accelerating force: RADIATIVE FORCE
- Lucy & Solomon (1970) - winds can be driven by absorption of radiation in spectral lines

Line-driven wind theory

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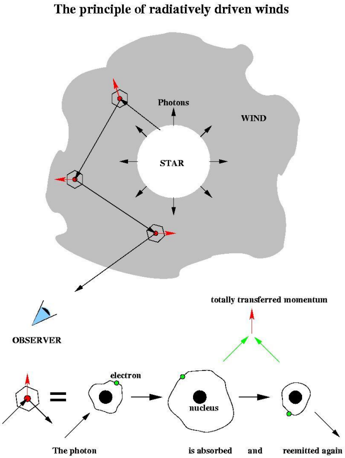
- Modern studies of hot stars' winds were stimulated mainly by UV observations
- Pioneering works of Lucy & Solomon (1970) and Castor, Abbott, & Klein (1975, CAK) serve as a basis for present hot star wind theory

Principle of radiative line-driving

Hot star winds are accelerated via a two-step process:

1. The photons are scattered in lines of ions of heavier elements (e.g., C, N, O, Ne, Si, P, S, Ni, Fe-group elements etc.)
   - physical process: momentum and energy transfer by absorption and scattering
2. The outward accelerated ions transfer their momenta to the bulk plasma of the wind (hydrogen and helium - mostly passive component)
   - physical process: Coulomb collisions

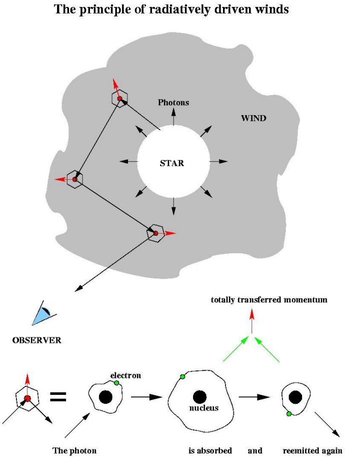
Principle of radiative line-driving



from homepage of Joachim Puls

1. The light scattering in lines of heavier elements
   - Photons transfer (part of) their *momentum* to heavier ions and electrons by line scattering
     - photon is absorbed by an ion

Principle of radiative line-driving



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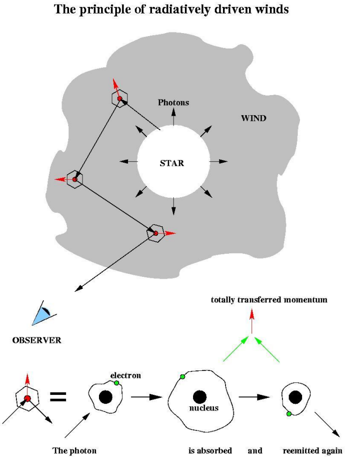

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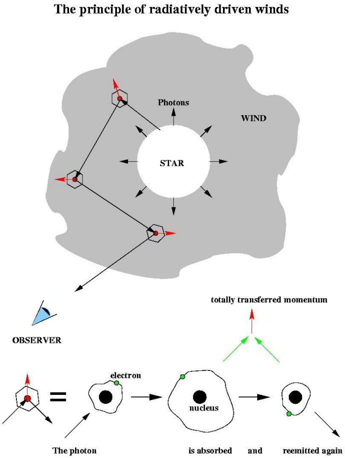

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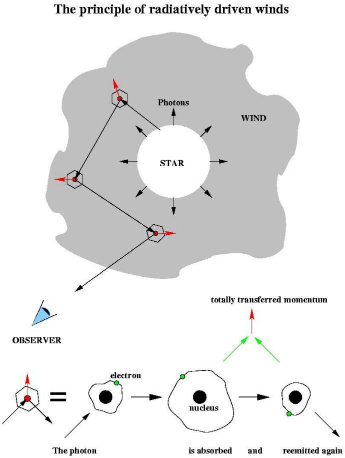

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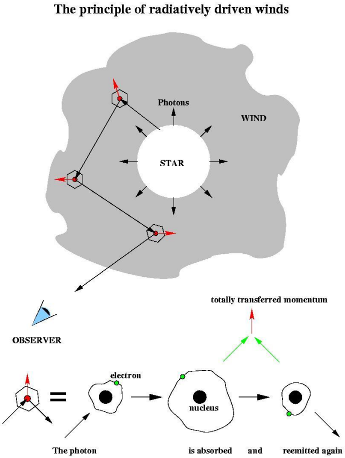

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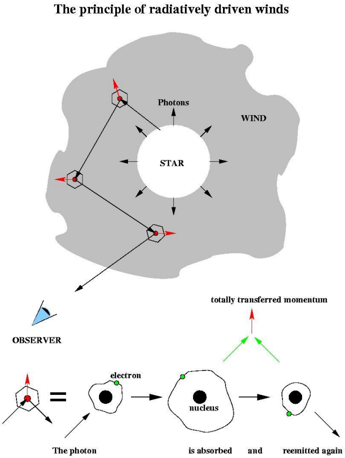

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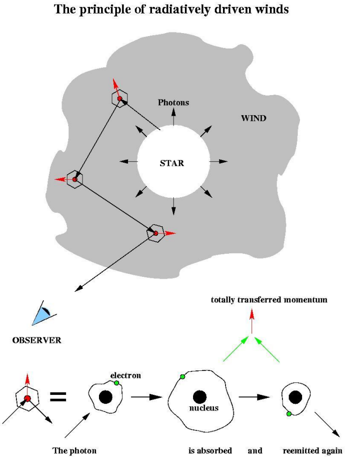

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     - only the outward directed acceleration due to absorption processes survives

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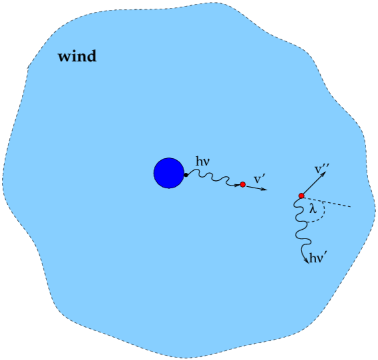
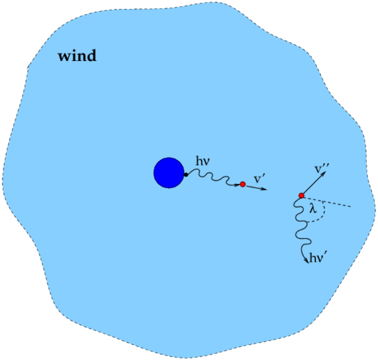


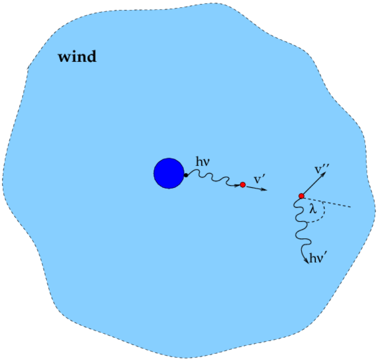
1. The light scattering in lines of heavier elements
   - momentum of an ion after absorption of photon
     $$mv_r' = mv_r + \frac{h\nu}{c}$$
   - increase of velocity
     $$\Delta v_r = \frac{h\nu}{c}$$
   - momentum of an ion after emission of photon
     $$mv_r'' = mv_r' - \frac{h\nu'}{c}\cos\lambda$$

Principle of radiative line-driving



1. The light scattering in lines of heavier elements
   - frequency of absorbed photon in observer frame
   $$\nu = \nu_0\,(1 + \frac{v_r}{c})$$
   - frequency of emitted photon in observer frame
   $$\nu' = \nu_0\,(1 + \frac{v'_r}{c})$$

Principle of radiative line-driving



1. The light scattering in lines of heavier elements
   - velocity of the ion after absorption and re-emission

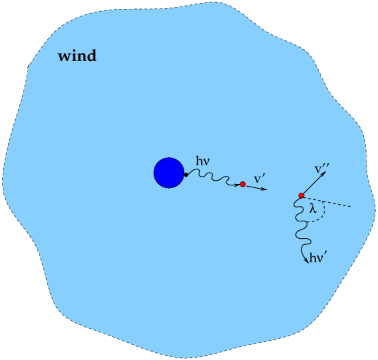
$$v_r'' = v_r + \frac{h\nu_0}{mc}\left(1 + \frac{v_r}{c}\right) - \frac{h\nu_0}{mc}\left(1 + \frac{v_r'}{c}\right)\cos\lambda$$

   - for $v \ll c$ and $h\nu_0 \ll c$

$$\Delta v_r = v_r'' - v_r = \frac{h\nu_0}{mc}(1 - \cos\lambda)$$

   - forward scattering $(\cos\lambda = 1) \Rightarrow$ the momentum does not increase
   - backward scattering $(\cos\lambda = -1) \Rightarrow$ the momentum increases by $2h\nu_0/c$
   - re-emission of photons is in random direction

Principle of radiative line-driving



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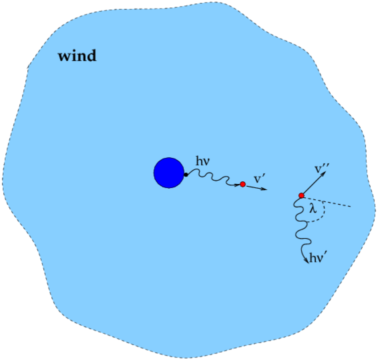
   - for $v \ll c$ and $h\nu_0 \ll c$

$$\Delta v_r = v_r'' - v_r = \frac{h\nu_0}{mc}(1 - \cos\lambda)$$

   - the mean transfer of momentum

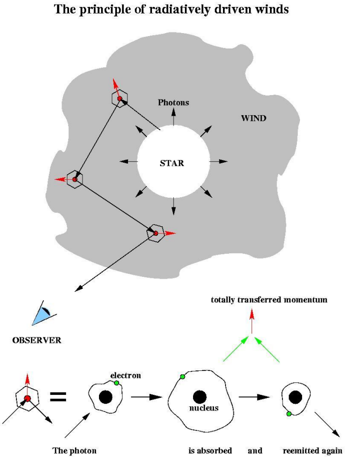
$$\langle m\Delta v\rangle = \frac{h\nu_0}{c}\frac{1}{4\pi}\int_{-\pi/2}^{\pi/2}(1 - \cos\lambda)\,2\pi\sin\lambda\,\mathrm{d}\lambda = \frac{h\nu_0}{c}$$

Principle of radiative line-driving



1. The light scattering in lines of heavier elements
   - line scatterings are of bound-bound type, i.e., line transitions
   - the wind acceleration is due to RADIATIVE LINE DRIVING

Principle of radiative line-driving



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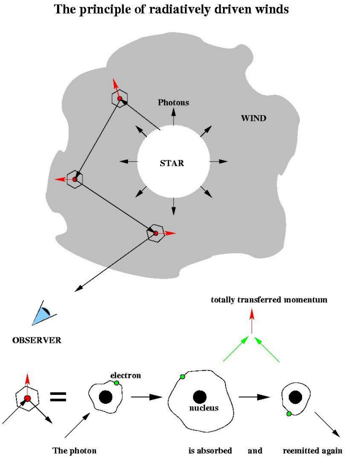
1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - the outward accelerated ions transfer their momenta to the bulk plasma of the wind (basically H and He) via Coulomb collisions

Principle of radiative line-driving



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   - the total wind is accelerated outward

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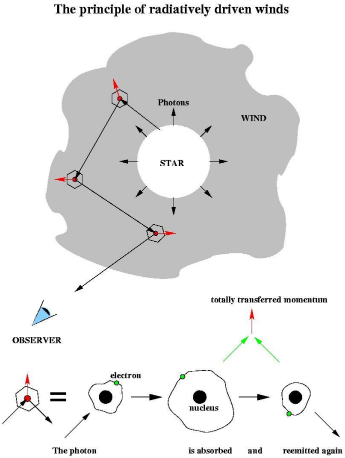

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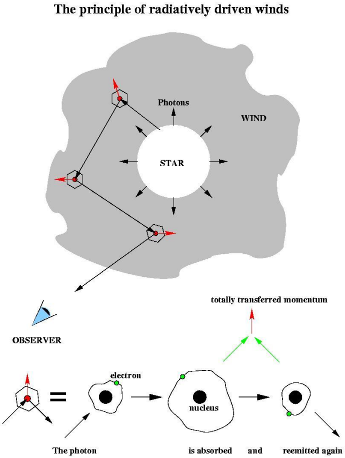
1. The light scattering in lines of heavier elements
2. Momentum transfer by Coulomb coupling
   - Condition for the Coulomb coupling to be efficient

$$t_{\mathrm{s}} < t_{\mathrm{d}}$$

- t_{s} [s] - characteristic time for slowing down heavier ions by collisions
- t_{d} [s] - time takes the heavier ions to gain a large drift velocity with respect to H and He
- first shown by Lucy and Solomon (1970) and improved by Lamers and Morton (1976)

Principle of radiative line-driving



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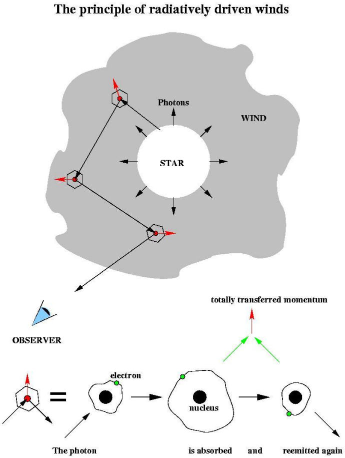
$$t_{\mathrm{s}} = 0.305\,\frac{A}{Z^2}\,\frac{T_e^{3/2}}{n_e(1-0.022\ln n_{\mathrm{e}})}$$

- A - mass of charged particles (in units of m_H)
- Z - charge (in units of the electron charge) due to interaction with H^+, He^{++} and electrons
- n_{e} - the electron density
- for winds with $10^8 \le n_e \le 10^{12} \;\Rightarrow\; (1-0.022\ln n_{\mathrm{e}}) \approx 0.5$

Principle of radiative line-driving



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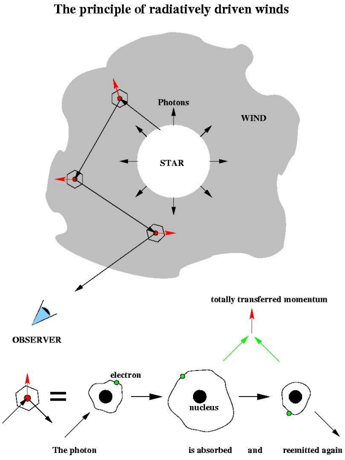
$$t_{\mathrm{d}} = \frac{v_{\mathrm{th}}}{g_i}$$

$$v_{\mathrm{th}} = \sqrt{\frac{2k_B T_{\mathrm{e}}}{m_H A_f}}$$

- A_f - atomic mass for field particles ($A_f \simeq 1$ for protons)
- g_i - acceleration of the absorbing ions
- T_{e} - temperature of the wind

Principle of radiative line-driving



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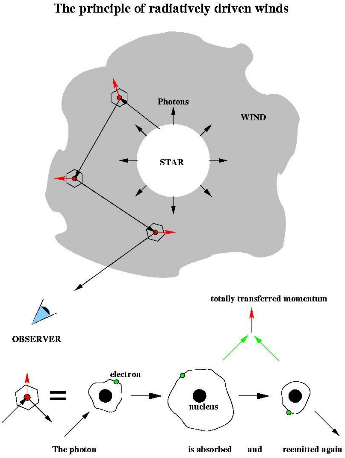
2. Momentum transfer by Coulomb coupling
   - Momentum transfer from photons to ions

$$\frac{\mathrm{d}(mv)}{\mathrm{d}t} = Am_H\, g_i = \frac{\pi e^2}{m_e} f \frac{\mathcal{F}_{\nu_0}}{c}$$

$$\mathcal{F}_{\nu_0} = \mathcal{F}^*_{\nu_0} \left(\frac{R_*}{r}\right)^2$$

- $(\pi e^2/m_e c)f$ - cross section for absorption
- $\mathcal{F}_{\nu_0}$ - flux at distance r from the star at the frequency of the line ν_0
- $\mathcal{F}^*_{\nu_0} = L^*_{\nu_0}/4\pi R^2_*$ - flux at surface of the star

Principle of radiative line-driving



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   - Condition for efficient Coulomb coupling

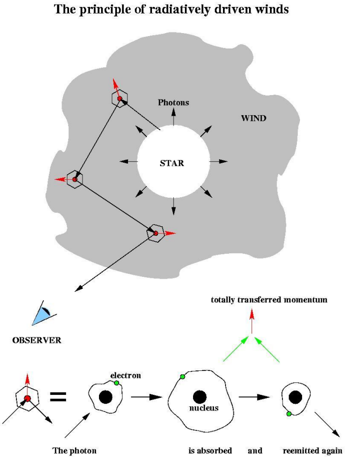
$$\frac{L^*_{\nu_0} T_\mathrm{e}}{4\pi r^2 n_\mathrm{e}} < \frac{Z^2 c}{0.61} \sqrt{2 k_B m_H} \left(\frac{\pi e^2}{m_e c} f\right)^{-1} A_f^{-1/2} = 3.6 \times 10^{-6} \qquad (1)$$

   - $A_f = 1$, $f = 0.1$, and $Z = 3$
   - $T_\mathrm{e} \simeq 0.5 T_\mathrm{eff}$, $L^*_{\nu_0} T_\mathrm{e} = 5.26 \times 10^{-12} L_*$; $n_\mathrm{e} = 5.2 \times 10^{23} gcm^{-3}$

$$\frac{L_* v}{\dot{M}} < 5.9 \times 10^{16}$$

   - for hot stars this is satisfied

Principle of radiative line-driving



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   - hydrogen and helium are mostly passive components of the wind (inefficient for wind driving)

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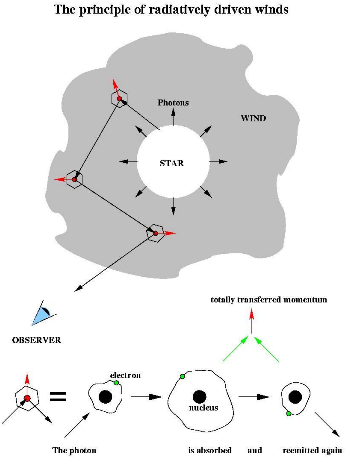

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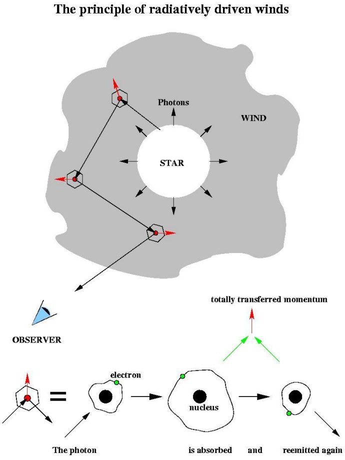
   - metal lines are responsible for the line driving

Principle of radiative line-driving



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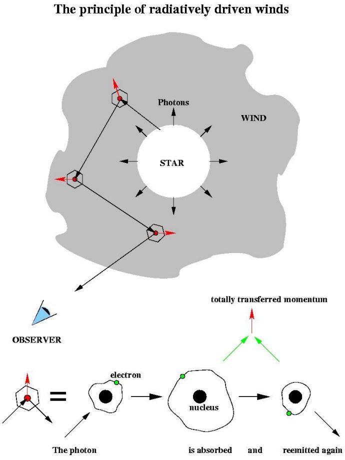
2. Momentum transfer by Coulomb coupling
   - hydrogen and helium are mostly passive components of the wind (inefficient for wind driving)
   - metal lines are responsible for the line driving
   - if transfer of momentum between metallic and passive wind component is efficient, the wind is well-coupled and can be treated as one component (Castor et al., 1976)

Principle of radiative line-driving



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   - hydrogen and helium are mostly passive components of the wind (inefficient for wind driving)
   - metal lines are responsible for the line driving
   - if transfer of momentum between metallic and passive wind component is efficient, the wind is well-coupled and can be treated as one component (Castor et al., 1976)
   - if the transfer of momentum is inefficient, the wind components may decouple (Springmann and Pauldrach, 1992, Krtička and Kubát 2000)

Wind hydrodynamic equations

Single-fluid treatment, neglecting viscosity and forces due to electric and magnetic fields

- equations of the continuity

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

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- equations of motion (momentum)

$$\frac{\partial v}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}_{\text{ex}}$$

  - $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{r}, \boldsymbol{t})$ - velocity field
  - $\rho = \rho(r, t)$ - mass density
  - $p = p(r, t)$ - gas pressure
  - $\boldsymbol{g}_{\text{ex}}$ - external acceleration; $\boldsymbol{g}_{\text{ex}} = \boldsymbol{g}_{\text{grav}} + \boldsymbol{g}_{\text{rad}}$

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  - $\boldsymbol{g}_{\text{ex}}$ - external acceleration; $\boldsymbol{g}_{\text{ex}} = \boldsymbol{g}_{\text{grav}} + \boldsymbol{g}_{\text{rad}}$
- energy equation
  - an approximate solution of the energy equation is allowed (see Klein and Castor, 1978)
  - T_{e} is approximately constant with radius and slightly less than T_{eff}, i.e. isothermal wind

Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of the continuity

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(\rho v_r r^2) = 0$$

- after integration $\Rightarrow$ total outward mass flux, i.e. $\dot{M}$

$$\dot{M} \equiv \frac{\mathrm{d}M_*}{\mathrm{d}t} = 4\pi\,\rho(r)\,v_r(r)\,r^2 = \text{const.}$$

Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of motion (momentum)

$$v_r \frac{\mathrm{d}v_r}{\mathrm{d}r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}r} - \boldsymbol{g}_{\text{grav}} + \boldsymbol{g}_{\text{rad}}$$

  - $g_{\text{grav}} = GM_*/r^2$ (G - the gravitational constant)
  - the gas pressure p is given by an ideal gas equation of state

$$p = \frac{\rho\, k_{\mathrm{B}} T}{\mu\, m_{\mathrm{H}}} = \rho\, a^2$$

  - a - isothermal speed of sound (const.)
  - k_{B} - Boltzmann's constant
  - m_{H} - the mass of a hydrogen atom
  - μ - the mean molecular weight of gas particles

Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of motion (momentum)

$$v_r \frac{\mathrm{d}v_r}{\mathrm{d}r} = -\frac{a^2}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{GM_*}{r^2} + \boldsymbol{g}_{\mathrm{rad}}$$

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Wind hydrodynamic equations

Assumption: stationary and spherically symmetric wind

- equations of motion (momentum)

$$\rho v_r \frac{\mathrm{d}v_r}{\mathrm{d}r} = -a^2 \frac{\mathrm{d}\rho}{\mathrm{d}r} - \frac{\rho G M_*}{r^2} + f_{\mathrm{rad}}$$

  - $f_{\mathrm{grav}} = \rho G M_* / r^2$ - gravitational force
  - f_{rad} - radiative force
  - the gas pressure p is given by an ideal gas equation of state
  - a - isothermal speed of sound (const.)

Radiative force

- f_{rad} - force due to a radiation field at a point r

$$f_{\text{rad}}(r) = \frac{1}{c} \int\limits_{\nu=0}^{\infty} \mathrm{d}\nu \oint\limits_{\Omega=4\pi} (\chi(r,\nu)\, I(r,\nu,\boldsymbol{k}) - \eta(r,\nu))\boldsymbol{k}\, \mathrm{d}\Omega$$

  - χ_ν - absorption coefficient
  - η_ν - emission coefficient
  - I_ν - radiative intensity
  - $\boldsymbol{k}$ - unit vector of the direction of the radiation propagation
- For isotropic emissivity, the integral over all angles vanishes as well as the second term, and $\chi(r,\nu)$ can be factored out of angular integration

Radiative force

- f_{rad} - force due to a radiation field at a point r

$$f_{\mathrm{rad}}(r) = \frac{1}{c}\int_{\nu=0}^{\infty}\chi(r,\nu)\,d\nu \oint_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,d\Omega = \frac{1}{c}\int_{\nu=0}^{\infty}\chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

  - χ_ν - absorption coefficient
  - I_ν - radiative intensity
  - $\boldsymbol{k}$ - unit vector of the direction of the radiation propagation
  - $\mathcal{F}$ - radiation flux

$$\mathcal{F}(r,\nu) = \oint_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,\mathrm{d}\Omega$$

Radiative force

- Total radiative force

$$f_{\text{rad}}(r) = f_{\text{cont}}(r) + f_{\text{line}}^{\text{tot}}(r)$$

  - $f_{\text{cont}}(r)$ - force due to continuum opacity
  - $f_{\text{line}}^{\text{tot}}(r)$ - force due to an ensemble of spectral lines
- Continuum opacity
  - continuum processes: atomic free-free and bound-free transitions and scattering on free electrons
  - continuum opacity due to free-free and bound-free processes can be neglected in the winds of O and B type stars
  - scattering of free electrons (Thomson scattering) - the main contributor to the continuum opacity

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c}\int_{\nu=0}^{\infty} \chi(r,\nu)\,d\nu \oint_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,d\Omega = \frac{1}{c}\int_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to radiation scattering on free electrons

$$f_{\mathrm{cont}}(r) = \frac{\chi_{\mathrm{th}}(r)}{c}\int_{\nu=0}^{\infty} \mathrm{d}\nu \oint_{\Omega=4\pi} I^{\mathrm{c}}(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,\mathrm{d}\Omega = \frac{\chi_{\mathrm{th}}(r)}{c}\int_{0}^{\infty} \mathcal{F}(r,\nu) = \frac{n_{\mathrm{e}}(r)\,\sigma_{\mathrm{Th}}\,L}{4\pi r^2 c}$$

  - I^{c} is "direct" continuum intensity from the stellar surface
  - χ_{th} - the Thomson scattering opacity

$$\chi_{\mathrm{th}}(r) = n_{\mathrm{e}}(r)\,\sigma_{\mathrm{Th}}$$

  - $\sigma_{\mathrm{Th}} = 6.65 \times 10^{-25}\,\mathrm{cm}^2$ - the cross-section for Thomson scattering
  - n_{e} - the number density of free electrons
  - $L = 4\pi r^2 \int_0^{\infty} \mathcal{F}(r,\nu)$

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int\limits_{\nu=0}^{\infty} \chi(r,\nu)\, d\nu \oint\limits_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\, \boldsymbol{k}\, d\Omega = \frac{1}{c} \int\limits_{\nu=0}^{\infty} \chi(r,\nu)\, \mathcal{F}(r,\nu)\, d\nu$$

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  - Ratio between the force due to the light scattering on free electrons and the gravitational force - Eddington factor (luminosity-to-mass ratio)

$$\Gamma_{\mathrm{e}} = \frac{f_{\mathrm{cont}}}{f_{\mathrm{grav}}} = \frac{\sigma_{\mathrm{Th}} \frac{n_{\mathrm{e}}}{\rho(r)} L}{4\pi c G M}$$

  - $\Gamma_{\mathrm{e}} \rightarrow 1$ - the Eddington limit

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c}\int_{\nu=0}^{\infty}\chi(r,\nu)\,d\nu \oint_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,d\Omega = \frac{1}{c}\int_{\nu=0}^{\infty}\chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to radiation scattering on free electrons

$$f_{\mathrm{cont}}(r) = \frac{\chi_{\mathrm{th}}(r)}{c}\int_{\nu=0}^{\infty}\mathrm{d}\nu \oint_{\Omega=4\pi} I^{\mathrm{c}}(r,\nu,\boldsymbol{k})\,\boldsymbol{k}\,\mathrm{d}\Omega = \frac{\chi_{\mathrm{th}}(r)}{c}\int_{0}^{\infty}\mathcal{F}(r,\nu) = \frac{n_{\mathrm{e}}(r)\,\sigma_{\mathrm{Th}}\,L}{4\pi r^2 c}$$

  - comparison with the gravity force

$$\Gamma_{\mathrm{e}} = \frac{f_{\mathrm{cont}}}{f_{\mathrm{grav}}} = \frac{\sigma_{\mathrm{Th}}\,\frac{n_{\mathrm{e}}}{\rho(r)}\,L}{4\pi c G M}$$

$$\Gamma_{\mathrm{e}} = 10^{-5}\left(\frac{L}{\mathrm{L}_{\odot}}\right)\left(\frac{M}{\mathrm{M}_{\odot}}\right)^{-1}$$

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r,\nu)\, d\nu \oint_{\Omega=4\pi} I(r,\nu,\boldsymbol{k})\, \boldsymbol{k}\, d\Omega = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r,\nu)\, \mathcal{F}(r,\nu)\, d\nu$$

- Radiative force due to radiation scattering on free electrons

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  - radiative force due to the light scattering on free electrons is important, but it never exceeds the gravity force

Radiative force

- Radiative force

$$f_{\text{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to line transition

$$\chi(r,\nu) = \frac{\pi e^2}{m_e c} \sum_{lines} g_i\, f_{ij} \left(\frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu)$$

  - $\phi_{ij}(\nu)$ - line profile; $\int_0^{\infty} \phi_{ij}(\nu)\mathrm{d}\nu = 1$
  - f_{ij} - oscillator strength
  - $n_i(r)$, $n_j(r)$ level occupation number
  - g_i - statistical weight of the level

Radiative force

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$$f_{\text{line}}(r) = \frac{\pi e^2}{m_e c^2} \sum_{lines} \int\limits_{0}^{\infty} g_i\, f_{ij} \left(\frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu)\,\mathcal{F}(r,\nu)\,\mathrm{d}\nu$$

  - lines influence on $\mathcal{F}(r,\nu)$
  - assumption: $\mathcal{F}(r,\nu)$ constant for frequencies corresponding to a given line, $\nu \approx \nu_{i,j}$

Radiative force

- Radiative force

$$f_{\text{rad}}(r) = \frac{1}{c}\int_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to line transition
  - maximum force

$$f_{\text{line}}^{max}(r) = \frac{\pi e^2}{m_e c^2}\sum_{lines} g_i\, f_{ij}\left(\frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j}\right)\mathcal{F}(r,\nu_{i,j})$$

  - $\nu_{i,j}$ - the line center frequency
  - neglect of $n_j(r) \ll n_i(r)$
  - $L_{\nu_{i,j}} = 4\pi r^2 \mathcal{F}(r,\nu_{i,j})$

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to line transition
  - maximum force: comparison with gravity

$$\frac{f_{\mathrm{line}}^{max}(r)}{f_{\mathrm{grav}}(r)} = \frac{L\,e^2}{4 m_e \rho\, G M\, c^2} \sum_{lines} f_{ij}\, n_i(r) \frac{L_{\nu_{i,j}}}{L}$$

  - $\nu_{i,j}$ - the line center frequency
  - neglect of $n_j(r) \ll n_i(r)$
  - $L_{\nu_{i,j}} = 4\pi r^2 \mathcal{F}(r,\nu_{i,j})$

Radiative force

- Radiative force

$$f_{\text{rad}}(r) = \frac{1}{c}\int\limits_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

- Radiative force due to line transition
  - maximum force: comparison with gravity

$$\frac{f_{\text{line}}^{max}(r)}{f_{\text{grav}}(r)} = \Gamma \sum_{\text{lines}} \frac{\sigma_{ij}}{\sigma_{\text{Th}}}\,\frac{n_i(r)}{n_e(r)}\,\frac{\nu_{i,j}\,L_\nu(\nu_{i,j})}{L}$$

$$\sigma_{ij} = \frac{\pi e^2\, f_{ij}}{\nu_{i,j}\, m_e c}$$

  - hydrogen: mostly ionised in the stellar envelopes $\Rightarrow$ $n_i(r)/n_e(r)$ very small $\Rightarrow$ negligible contribution to radiative force
  - neutral helium: $n_i(r)/n_e(r)$ very small $\Rightarrow$ negligible contribution to radiative force
  - ionised helium: very small contribution to the radiative force

Radiative force

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- Radiative force due to line transition
  - maximum force: comparison with gravity

$$\frac{f_{\text{line}}^{max}(r)}{f_{\text{grav}}(r)} = \Gamma \sum_{\text{lines}} \frac{\sigma_{ij}}{\sigma_{\text{Th}}} \frac{n_i(r)}{n_e(r)} \frac{\nu_{i,j}\, L_\nu(\nu_{i,j})}{L}$$

$$\sigma_{ij} = \frac{\pi e^2 f_{ij}}{\nu_{i,j}\, m_e c}$$

  - heavier elements (Fe, C, N, O, . . .): large number of lines, $\sigma_{ij}/\sigma_{\text{Th}} \approx 10^7 \Rightarrow f_{\text{line}}^{max}/f_{\text{grav}}$ up to 10^3
  - radiative force may be larger than gravity (for many O stars $f_{\text{line}}^{max}/f_{\text{grav}} \approx 2000$, Abbott 1982, Gayley 1995) $\Rightarrow$ stellar wind

Radiative force

- Radiative force

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int\limits_{\nu=0}^{\infty} \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

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$$f_{\mathrm{line}}(r) = \frac{\pi e^2}{m_e c^2} \sum_{lines} \int\limits_0^{\infty} g_i\, f_{ij} \left(\frac{n_i(r)}{g_i} - \frac{n_j(r)}{g_j} \right) \phi_{ij}(\nu)\,\mathcal{F}(r,\nu)\,\mathrm{d}\nu$$

  - the main problem: the line opacity (lines may be optically thick) $\Rightarrow$
  - necessary to solve the radiative transfer equation

Sobolev approximation

- Sobolev (1947) developed approach for treating line scattering in a rapidly accelerating flow
- This approximation is valid only if the velocity gradient is sufficiently large
- Due to the Doppler shift, the geometrical size in which a line can absorb photons with the fixed frequency is so small that χ_L and ρ change very little
- The profile function can be approximated with a δ-function that is sharply peaked around the central line frequency
- "Sobolev length"

$$L_{\mathrm{S}} \equiv \frac{v_{\mathrm{th}}}{\mathrm{d}v/\mathrm{d}r} \ll H \equiv \frac{\rho}{\mathrm{d}\rho/\mathrm{d}r} \approx \frac{v}{\mathrm{d}v/\mathrm{d}r}$$

  - H - a typical flow variation scale
  - $\rho/(\mathrm{d}\rho/\mathrm{d}r)$ and $v/(\mathrm{d}v/\mathrm{d}r)$ - the density and velocity scale length
  - simplification of the calculation of f_{line} possible

The radiative transfer equation

Assumptions: spherical symmetry, stationary (time-independent) flow

$$\mu \frac{\partial}{\partial r} I(r,\mu,\nu) + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} I(r,\mu,\nu) = \eta(r,\mu,\nu) - \chi(r,\mu,\nu)\, I(r,\mu,\nu)$$

- frame of static observer
- $\mu = \cos\theta$
- $I(r,\mu,\nu)$ - specific intensity
- $\chi(r,\mu,\nu)$ - absorption (extinction) coefficient
- $\eta(r,\mu,\nu)$ - emissivity (emission coefficient)
- problem: $\chi(r,\mu,\nu)$ and $\eta(r,\mu,\nu)$ depend on μ due to the Doppler effect
- solution: use comoving-frame (CMF)

CMF radiative transfer equation

Assumptions: spherical symmetry, stationary (time-independent) flow

$$\mu\frac{\partial}{\partial r}I(r,\mu,\nu)+\frac{1-\mu^2}{r}\frac{\partial}{\partial\mu}I(r,\mu,\nu)-\frac{\nu\,v(r)}{c\,r}\left(1-\mu^2+\frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)\frac{\partial}{\partial\nu}I(r,\mu,\nu)=\eta(r,\nu)-\chi(r,\nu)\,I(r,\mu,\nu)$$

- $\chi(r,\mu,\nu)$ and $\eta(r,\mu,\nu)$ do not depend on μ
- neglected aberration, advection (unimportant for $v \ll c$)

CMF radiative transfer equation

The Sobolev transfer equation (Castor 2004)

$$\cancel{\mu \frac{\partial}{\partial r} I(r,\mu,\nu) + \frac{1-\mu^2}{r} \frac{\partial}{\partial \mu} I(r,\mu,\nu)} - \frac{\nu\, v(r)}{c\, r} \left(1-\mu^2 + \frac{\mu^2 r}{v(r)} \frac{\mathrm{d}v(r)}{\mathrm{d}r}\right) \frac{\partial}{\partial \nu} I(r,\mu,\nu) =$$
$$\eta(r,\nu) - \chi(r,\nu)\, I(r,\mu,\nu)$$

- possible when $\frac{\nu\, v(r)}{c\, r} \frac{\partial}{\partial \nu} I(r,\mu,\nu) \gg \frac{\partial}{\partial r} I(r,\mu,\nu)$

CMF radiative transfer equation

Solution of the transfer equation for one line

$$-\frac{\nu\, v(r)}{c\, r}\left(1-\mu^2+\frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)\frac{\partial}{\partial\nu}I(r,\mu,\nu)=\eta(r,\nu)-\chi(r,\nu)\,I(r,\mu,\nu)$$

- line absorption and emission coefficients

$$\chi(r,\nu)=\frac{\pi e^2}{m_e c}\,g_i\,f_{ij}\left(\frac{n_i(r)}{g_i}-\frac{n_j(r)}{g_j}\right)\phi_{ij}(\nu)=\chi_L(r)\,\phi_{ij}(\nu)$$

$$\eta(r,\nu)=\frac{2h\nu^3}{c^2}\frac{\pi e^2}{m_e c}g_i\,f_{ij}\,\frac{n_j(r)}{g_j}\,\phi_{ij}(\nu)=\chi_L(r)\,S_L(r)\,\phi_{ij}(\nu)$$

$$\chi_L(r)=\frac{\pi e^2}{m_e c}\,g_i\,f_{ij}\left(\frac{n_i(r)}{g_i}-\frac{n_j(r)}{g_j}\right)$$

CMF radiative transfer equation

Solution of the transfer equation for one line

$$-\frac{\nu\, v(r)}{c\, r}\left(1-\mu^2+\frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)\frac{\partial}{\partial\nu}I(r,\mu,\nu)=\chi_L(r)\,\phi_{ij}(\nu)(S_L(r)-I(r,\mu,\nu))$$

- introduce a new variable

$$y=\int_{\nu}^{\infty}\phi_{ij}(\nu')\mathrm{d}\nu'$$

- where
  - y = 0: the incoming side of the line
  - y = 1: the outgoing side of the line

CMF radiative transfer equation

Solution of the transfer equation for one line

$$-\frac{\nu\, v(r)}{c\, r}\left(1-\mu^2+\frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)\frac{\partial}{\partial y}I(r,\mu,y)=\chi_L(r)\,\phi_{ij}(\nu)(S_L(r)-I(r,\mu,y))$$

- assumptions:
  - variables do not significantly vary with r within the “resonance zone” $\Rightarrow$
  - fixed r, $\frac{\partial}{\partial y} \to \frac{\mathrm{d}}{\mathrm{d}y}$
  - $\nu \to \nu_0$
- integration possible

CMF radiative transfer equation

Solution of the transfer equation for one line

$$I(y) = I_c(\mu)e^{-\tau(\mu)y} + S_L 1 - e^{-\tau(\mu)y}$$

- the Sobolev optical depth in spherical symmetry

$$\tau(\mu) = \frac{\chi_L(r)cr}{\nu_0 v(r)\left(1 - \mu^2 + \frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)}$$

- the boundary condition is $I(y = 0) = I_c(\mu)$
- τ is given by the slope $\Rightarrow$ $\tau \sim \left(\frac{\mathrm{d}v}{\mathrm{d}r}\right)^{-1}$

Radiative force

the radial component; force per unit of volume

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int_0^\infty \chi(r,\nu)\,\mathcal{F}(r,\nu)\,d\nu$$

$$f_{\mathrm{rad}}(r) = \frac{1}{c} \int_0^\infty \chi(r,\nu)\,\mathrm{d}\nu \oint I(r,\nu,\boldsymbol{k})\boldsymbol{k}\,\mathrm{d}\Omega$$

$$f_{\mathrm{rad}}(r) = \frac{2\pi}{c} \int_0^\infty \chi_L(r)\,\phi_{ij}(\nu)\,\mathrm{d}\nu \int_{-1}^{1} \mu\, I(r,\mu,\nu)\,\mathrm{d}\mu$$

$$f_{\mathrm{rad}}(r) = \frac{2\pi\,\chi_L(r)}{c} \int_0^1 \mathrm{d}y \int_{-1}^{1} \mu\, I(r,\mu,\nu)\,\mathrm{d}\mu$$

Radiative force

the radial component; force per unit of volume

$$f_{\mathrm{rad}}(r) = \frac{2\pi \chi_L(r)}{c} \int_0^1 \mathrm{d}y \int_{-1}^{1} \left[I_c(\mu)\, e^{-\tau(\mu)y} + S_L \left(1 - e^{-\tau(\mu)y}\right) \right] \mu \, \mathrm{d}\mu$$

- where the Sobolev optical depth is

$$\tau(\mu) = \frac{\chi_L(r) c r}{\nu_0 v(r) \left(1 - \mu^2 + \frac{\mu^2 r}{v(r)} \frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)}$$

- no net contribution of the emission to the radiative force (S_L is isotropic in the CMF)

$$f_{\mathrm{rad}}(r) = \frac{2\pi \chi_L(r)}{c} \int_0^1 \mathrm{d}y \int_{-1}^{1} \mu \, I_c(\mu)\, e^{-\tau(\mu)y} \mathrm{d}\mu$$

Radiative force

the radial component; force per unit of volume

$$f_{\mathrm{rad}}(r) = \frac{2\pi\chi_L(r)}{c} \int_{-1}^{1} \mu\, I_c(\mu)\, \frac{1 - e^{-\tau(\mu)y}}{\tau(\mu)} \mathrm{d}\mu$$

- inserting

$$\tau(\mu) = \frac{\chi_L(r)cr}{\nu_0 v(r)\left(1 - \mu^2 + \frac{\mu^2 r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r}\right)}$$

$$f_{\mathrm{rad}}(r) = \frac{2\pi\,\nu_0\, v(r)}{r\, c^2} \int_{-1}^{1} \mu\, I_c(\mu) \left[1 + \mu^2\sigma(r)\right] \left\{1 - exp\left[-\frac{\chi_L(r)\, cr}{\nu_0 v(r)\,(1 + \mu^2\sigma(r))}\right]\right\} \mathrm{d}\mu$$

$$\sigma(r) = \frac{r}{v(r)}\frac{\mathrm{d}v(r)}{\mathrm{d}r} - 1$$

Sobolev (1957), Castor (1974), Rybicki & Hummer (1978)

Radiative force

- Optically thin line

$$f_{\text{rad}}(r) = \frac{2\pi\,\nu_0\,v(r)}{r\,c^2}\int_{-1}^{1}\mu\,I_c(\mu)\left[1+\mu^2\sigma(r)\right]\left\{1 - exp\left[-\frac{\chi_L(r)\,cr}{\nu_0 v(r)\left(1+\mu^2\sigma(r)\right)}\right]\right\}\mathrm{d}\mu$$

  - Optically thin line

$$\frac{\chi_L(r)\,cr}{\nu_0 v(r)\left(1+\mu^2\sigma(r)\right)} \ll 1$$

$$f_{\text{rad}}(r) \sim 1 - exp\left[-\frac{\chi_L(r)\,cr}{\nu_0 v(r)\left(1+\mu^2\sigma(r)\right)}\right] \approx \frac{\chi_L(r)\,cr}{\nu_0 v(r)\left(1+\mu^2\sigma(r)\right)}$$

$$f_{\text{rad}}(r) = \frac{2\pi}{c}\int_{-1}^{1}\mu\,I_c(\mu)\,\chi_L(r)\,\mathrm{d}\mu$$

Radiative force

- Optically thin line

$$f_{\mathrm{rad}}(r) = \frac{2\pi\,\nu_0\,v(r)}{r\,c^2}\int_{-1}^{1}\mu\,I_c(\mu)\left[1+\mu^2\sigma(r)\right]\left\{1-exp\left[-\frac{\chi_L(r)\,cr}{\nu_0 v(r)\,(1+\mu^2\sigma(r))}\right]\right\}\mathrm{d}\mu$$

$$f_{\mathrm{rad}}(r) = \frac{2\pi}{c}\int_{-1}^{1}\mu\,I_c(\mu)\,\chi_L(r)\,\mathrm{d}\mu$$

$$f_{\mathrm{rad}}(r) = \frac{1}{c}\chi_L(r)\,\mathcal{F}(r)$$

- optically thin radiative force proportional to the radiative flux $\mathcal{F}(r)$
- optically thin radiative force proportional to the normalised line opacity $\chi_L(r)$ (or to the density)
- the same result as for the static medium

Radiative force

- Optically thick line

$$f_{\mathrm{rad}}(r) = \frac{2\pi\,\nu_0\,v(r)}{r\,c^2} \int_{-1}^{1} \mu\, I_c(\mu)\left[1+\mu^2\sigma(r)\right]\left\{1 - exp\left[-\frac{\chi_L(r)\,cr}{\nu_0 v(r)\,(1+\mu^2\sigma(r))}\right]\right\} \mathrm{d}\mu$$

$$\frac{\chi_L(r)\,cr}{\nu_0 v(r)\,(1+\mu^2\sigma(r))} \gg 1$$

$$f_{\mathrm{rad}}(r) \sim 1 - exp\left[-\frac{\chi_L(r)\,cr}{\nu_0 v(r)\,(1+\mu^2\sigma(r))}\right] \approx 1$$

$$f_{\mathrm{rad}}(r) = \frac{2\pi\,\nu_0\,v(r)}{r\,c^2} \int_{-1}^{1} \mu\, I_c(\mu)\left[1+\mu^2\sigma(r)\right] \mathrm{d}\mu$$

Radiative force

- Optically thick line

$$f_{\text{rad}}(r) = \frac{2\pi\,\nu_0\,v(r)}{r\,c^2} \int_{-1}^{1} \mu\, I_c(\mu)\left[1 + \mu^2\sigma(r)\right] \mathrm{d}\mu$$

  - neglect of the limb darkening:
  - $\mu_* = \sqrt{1 - \frac{R_*}{r^2}}$

$$I_c(\mu) = \begin{cases} I_c = const. & \mu \geq \mu_*, \\ 0, & \mu < \mu_* \end{cases}$$

- $\mathcal{F} = 2\pi \int_{\mu_*}^{1} \mu\, I_c\, \mathrm{d}\mu = \pi\, \frac{R_*}{r^2} I_c$

$$f_{\text{rad}}(r) = \frac{\nu_0\, v(r)\, \mathcal{F}(r)}{r\,c^2}\left[1 + \sigma(r)\left(1 - \frac{1}{2}\frac{R_*}{r^2}\right)\right]$$

Radiative force

- Optically thick line

$$f_{\text{rad}}(r) = \frac{\nu_0\, v(r)\, \mathcal{F}(r)}{r\, c^2} \left[1 + \sigma(r)\left(1 - \frac{1}{2}\frac{R_*}{r^2}\right)\right]$$

$$\sigma(r) = \frac{r}{v(r)} \frac{\mathrm{d}v(r)}{\mathrm{d}r} - 1$$

- large distance from the star: $r \gg R_*$

$$f_{\text{rad}}(r) = \frac{\nu_0\, \mathcal{F}(r)}{c^2} \frac{\mathrm{d}v(r)}{\mathrm{d}r}$$

- optically thick radiative force proportional to the radiative flux $\mathcal{F}(r)$
- optically thick radiative force proportional to $\mathrm{d}v(r)/\mathrm{d}r$
- optically thick radiative force does not depend on the level populations (opacity) or the density