Magneto-rotational instability

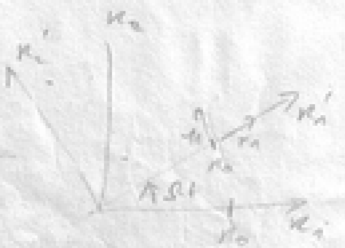
EP 34 (MRI 1)

- nejprve mechanická analogie: dvě částice v centrálním poli spojené pružinou tuhosti K

pohyb. rovnice v korotujícím syst. ($x = R' - R_0$)

$$\frac{d^2x_1}{dt^2} = 2\Omega\frac{dy_1}{dt} - R\frac{d\Omega^2}{dR}x_1 + \frac{K}{m}(x_2 - x_1) \qquad (1)$$

$$\frac{d^2y_1}{dt^2} = -2\Omega\frac{dx_1}{dt} + \frac{K}{m}(y_2 - y_1) \qquad (2)$$



$$\frac{d^2x_2}{dt^2} = 2\Omega\frac{dy_2}{dt} - R\frac{d\Omega^2}{dR}x_2 + \frac{K}{m}(x_1 - x_2) \qquad (3)$$

$$\frac{d^2y_2}{dt^2} = -2\Omega\frac{dx_2}{dt} + \frac{K}{m}(y_1 - y_2) \qquad (4)$$

↑
coriolis

$-R\frac{d\Omega^2}{dR}x_1$ = grav. + odstředivá síla = $(\Omega^2 - \Omega_K^2)\vec{R}$

Ω = konstanta = $\Omega_K(R)$

exp. přepsal $\Omega \to \Omega_0 = \Omega_K(R_0)$, Ω funkce $\to \Omega_K(R)$

(1)+(3): $\frac{d^2}{dt^2}(x_1 + x_2) = 2\Omega\frac{d}{dt}(y_1 + y_2) - R\frac{d\Omega^2}{dR}(x_1 + x_2)$

(2)+(4): $\frac{d^2}{dt^2}(y_1 + y_2) = -2\Omega\frac{d}{dt}(x_1 + x_2)$

řeš. ve tvaru $\propto \exp(-i\omega t)$ $\to$ $\omega^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 \equiv \varkappa^2$

(1)−(3): $\frac{d^2}{dt^2}(x_1 - x_2) = 2\Omega\frac{d}{dt}(y_1 - y_2) - R\frac{d\Omega^2}{dR}(x_1 - x_2) - \frac{2K}{m}(x_1 - x_2)$

(2)−(4): $\frac{d^2}{dt^2}(y_1 - y_2) = -2\Omega\frac{d}{dt}(x_1 - x_2) - \frac{2K}{m}(y_1 - y_2)$

→ další řešení:

$$\omega^4 - \left(\frac{4K}{m} + \varkappa^2\right)\omega^2 + \frac{2K}{m}\left(\frac{2K}{m} + R\frac{d\Omega^2}{dR}\right) = 0$$

- imaginární kořeny pro $\left(\frac{2K}{m} + R\frac{d\Omega^2}{dR}\right) < 0 \wedge K > 0$

jinak by dva kořeny byly $= 0$

- protože v běžných discích je $\frac{d\Omega}{dR} < 0$,

<u>dostatečně slabá</u> pružina vede k nestabilitě

EP35 (MRI 2)

- pružinu nahradíme mg. polem
- maximálně si zjednodušíme život

→ $\vec{B} = B_z \vec{e}_z + \vec{B}_1$, $\nabla\cdot v = 0$, $\vec{k} = k\vec{e}_z$, $\vec{v} = \vec{v}_\varphi + \vec{v}_1$

- nestlačitelnost + $\vec{k} = k\vec{e}_z$ ⇒ $v_{1z} = 0$
- Maxwell + $\vec{k} = k\vec{e}_z$ ⇒ $B_{1z} = 0$

- ideální MHD → zamrznutí mg. pole do plazmatu:

$$\frac{\partial B}{\partial t} = \nabla\times(v\times B)$$

- v cylindrických souř.

$$\nabla\times A = \left(\frac{1}{R}\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z},\ \frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R},\ \frac{1}{R}\frac{\partial}{\partial R}(RF_\varphi) - \frac{1}{R}\frac{\partial F_R}{\partial\varphi}\right)$$

⊕ $\frac{\partial}{\partial\varphi} = 0$ ⊕ zanedbání $O(B_1^2)$, $O(B_1 v_1)$:

$$\frac{\partial B_{1R}}{\partial t} = B_z \frac{\partial v_{1R}}{\partial z} \qquad (5)$$

$$\frac{\partial B_{1\varphi}}{\partial t} = R\frac{d\Omega}{dR} B_{1R} + B_z \frac{\partial v_{1\varphi}}{\partial z} \qquad (6)$$

- mě tam ve druhé rovnici vycházelo $\frac{\partial v_\varphi}{\partial R}$ namísto $R\frac{d\Omega}{dR}$.

- Eulerovy rovnice s Lorentzovou silou $\vec{f} = \vec{j}\times\vec{B}$

⊕ Maxwell: $\nabla\times B = \mu_0 j$

$$\frac{\partial v_{1R}}{\partial t} - 2\Omega v_{1\varphi} = \frac{B_z}{\mu_0\rho}\frac{\partial B_{1R}}{\partial z} \qquad (7)$$

→

$$\frac{\partial v_{1\varphi}}{\partial t} + \frac{\kappa^2}{2\Omega} v_{1R} = \frac{B_z}{\mu_0\rho}\frac{\partial B_{1\varphi}}{\partial z} \qquad (8)$$

$$0 = -\frac{\partial P_1}{\partial z} - \frac{B_{1\varphi}}{\mu_0\rho}\frac{\partial B_{1\varphi}}{\partial z}$$ ⇐ tuhle nebudeme potřebovat

poruchy $\propto \exp[i(kz - \omega t)]$

Alfvénova rychlost: $v_A^2 = \frac{B_z^2}{\mu_0\rho}$

→ $$\omega^4 - (2k^2v_A^2 + \kappa^2)\omega^2 + k^2v_A^2\left(k^2v_A^2 + R\frac{d\Omega^2}{dR}\right) = 0$$

→ nestability

- maximální růst pro $\omega = \frac{1}{2}R\left|\frac{d\Omega}{dR}\right|$, $kv_A = \Omega^2 - \frac{\kappa^4}{16\Omega^2}$

- viskozita způsobená MRI by se měla projevit jako člen $\propto B_R B_\varphi$ v $\Pi_{R\varphi}$

není mi jasné, jestli by do toho šla transformovat pravá strana rce (8) ad rovnice (5)–(8):

$$\nabla \times (v \times B) = v(\nabla \cdot B) + (B\cdot\nabla)v - B(\nabla\cdot v) - (v\cdot\nabla)B$$

kovariantně:

$$[\nabla\times(U\times B)]^a = U^a B^b{}_{;b} + B^b U^a{}_{;b} - B^a U^b{}_{;b} - U^b B^a{}_{;b}$$

kde $U^a = (\dot{R}, \dot{\varphi}, \dot{z}) = (v_R, \frac{v_\varphi}{R}, v_z)$

$$g_{ab} = \begin{pmatrix} 1 & & \\ & R^2 & 0 \\ 0 & & 1 \end{pmatrix} \qquad \Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}\left(g_{cd,b} + g_{bd,c} - g_{bc,d}\right)$$

jediné nenulové: $\Gamma^\varphi{}_{R\varphi} = \Gamma^\varphi{}_{\varphi R} = \frac{1}{R}$, $\Gamma^R{}_{\varphi\varphi} = -R$

$B^b{}_{;b} = 0$ (Maxwell), $U^b{}_{;b} = 0$ – nestlačitelnost

$$[\nabla\times(U\times B)]^R = B^b U^R{}_{;b} - U^b B^R{}_{;b} =$$

$$= B^R U^R{}_{,R} + B^\varphi U^R{}_{,\varphi} - \cancel{B^\varphi R U^\varphi} + B^z U^R{}_{,z} - U^R B^R{}_{,R} - U^\varphi B^R{}_{,\varphi} + \cancel{U^\varphi R B^\varphi} - U^z B^R{}_{,z} =$$

$$= B^z U^R{}_{,z} + O(B_1^2)$$

$$[\nabla\times(U\times B)]^\varphi = \frac{1}{R}[\nabla\times(v\times B)]_\varphi =$$

$$= B^R U^\varphi{}_{;R} + B^\varphi U^\varphi{}_{;\varphi} + B^z U^\varphi{}_{;z} - U^R B^\varphi{}_{;R} - U^\varphi B^\varphi{}_{;\varphi} - U^z B^\varphi{}_{;z} =$$

$$= B^R U^\varphi{}_{,R} + \cancel{\tfrac{1}{R} B^R U^\varphi} + B^\varphi U^\varphi{}_{,\varphi} + \cancel{B^\varphi \tfrac{1}{R} U^R} + B^z U^\varphi{}_{,z} -$$

$$- U^R B^\varphi{}_{,R} - \cancel{\tfrac{1}{R} U^R B^\varphi} - U^\varphi B^\varphi{}_{,\varphi} - \cancel{U^\varphi \tfrac{1}{R} B^R} - U^z B^\varphi{}_{,z} =$$

$$= B^R \frac{d\Omega}{dR} + B^z \frac{\partial\Omega}{\partial z} + O(B_1^2).$$

tento člen se ruší s gravitačním, takže by bylo na pravé straně „vyrovnání" rce (7)

$$U\cdot\nabla U = U^a U^b{}_{;a}$$

$$[U\cdot\nabla U]^R = U^R U^R{}_{,R} + U^\varphi U^R{}_{,\varphi} - U^\varphi R U^\varphi + U^z U^R{}_{,z} =$$

$$= -(U^\varphi)^2 R + O(U_1^2) = -\frac{1}{R} v_\varphi^2 = -R\Omega^2 - 2\Omega v_{1\varphi} + O(v_1^2)$$

$= 0 \Leftarrow v_R = 0$

$$[U\cdot\nabla U]^\varphi = U^R U^\varphi{}_{,R} + \frac{1}{R} U^R U^\varphi + U^\varphi U^\varphi{}_{,R} + \frac{1}{R} U^\varphi U^R + U^z U^\varphi{}_{,z} =$$

$$= U^R \frac{d\Omega}{dR} + \frac{2}{R} U^R \Omega = \frac{1}{R}\frac{U^R}{2\Omega}\left[2R\Omega\frac{d\Omega}{dR} + 4\Omega^2\right] = \frac{1}{R}\frac{\varkappa^2}{2\Omega} U^R$$

$$= \frac{1}{R}[v\cdot\nabla v]_\varphi$$

ad MR1

$\nabla \times (v \times B) = \frac{\partial B}{\partial t}$ $\quad k = k_z e_z \quad B_{1z} = 0 \quad v_{1z} = 0$

(R, φ, z)

$[\nabla \times (v \times B)]_R = \frac{1}{R}\frac{\partial}{\partial \varphi}[v \times B]_z - \frac{\partial}{\partial z}[v \times B]_\varphi =$

$= \underbrace{\frac{1}{R}\frac{\partial}{\partial \varphi}[v_{1R} B_\varphi - v_\varphi B_{1R}]}_{=0} - \frac{\partial}{\partial z}[\underbrace{v_{1z}}_{=0} B_R - v_{1R} B_z] \approx B_z \frac{\partial v_{1R}}{\partial z}$

tady to asi bude s tou derivací hodně jinak

$\frac{\partial B_\varphi}{\partial t} = [\nabla \times (v \times B)]_\varphi = \frac{\partial}{\partial z}[v \times B]_R - \frac{\partial}{\partial R}[v \times B]_z =$

$= \frac{\partial}{\partial z}(v_\varphi B_z - v_z B_\varphi) - \frac{\partial}{\partial R}(v_R B_\varphi - v_\varphi B_R)$

nedaří se vyjít ??

$$\frac{\partial B_\varphi}{\partial t} = B_z \frac{\partial v_\varphi}{\partial z} + B_R R \frac{\partial \Omega}{\partial R}$$

$f_{Lorentz} = j \times B = \frac{1}{\mu_0}(\nabla \times B) \times B$

$f_R = \frac{1}{\mu_0}[\nabla \times B]_\varphi \times B_z - \frac{1}{\mu_0}[\nabla \times B]_z \times B_\varphi =$

$= \frac{1}{\mu_0}\left(\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right) B_z - \frac{1}{\mu_0}\left(\frac{1}{R}\frac{\partial}{\partial R}(R B_\varphi) - \frac{1}{R}\frac{\partial B_R}{\partial \varphi}\right) B_\varphi$

$\approx \frac{1}{\mu_0} B_z \frac{\partial B_R}{\partial z} + O(B_1^2)$

$f_\varphi = \frac{1}{\mu_0}[\nabla \times B]_z B_R - \frac{1}{\mu_0}[\nabla \times B]_R B_z = \frac{1}{\mu_0}\left(\frac{1}{R}\frac{\partial}{\partial R}(R B_\varphi) - \frac{1}{R}\frac{\partial B_R}{\partial \varphi}\right) B_R$

$- \frac{1}{\mu_0}\left(\frac{1}{R}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right) B_z \approx \frac{1}{\mu_0} B_z \frac{\partial B_\varphi}{\partial z} + O(B_1^2)$

$f_z = \frac{1}{\mu_0}[\nabla \times B]_R B_\varphi - \frac{1}{\mu_0}[\nabla \times B]_\varphi B_R = \frac{1}{\mu_0}\left(\frac{1}{R}\frac{\partial B_z}{\partial \varphi} - \frac{\partial B_\varphi}{\partial z}\right) B_\varphi -$

$- \frac{1}{\mu_0}\left[\frac{\partial B_R}{\partial z} - \frac{\partial B_z}{\partial R}\right] B_R \approx - \frac{1}{\mu_0} B_\varphi \frac{\partial B_\varphi}{\partial z} + O(B_1^2)$

$\nabla \times F = \left(\frac{1}{R}\frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z},\ \frac{\partial F_R}{\partial z} - \frac{\partial F_z}{\partial R},\ \frac{1}{R}\frac{\partial}{\partial R}(R F_\varphi) - \frac{1}{R}\frac{\partial F_R}{\partial \varphi}\right)$

$R \quad \varphi \quad z$

$$-\omega^2(x_1+x_2) = -2i\omega\Omega(y_1+y_2) - R\frac{d\Omega^2}{dR}(x_1+x_2)$$

$$-\omega^2(y_1+y_2) = 2i\omega\Omega(x_1+x_2)$$

$$\left(R\frac{d\Omega^2}{dR} - \omega^2\right)(x_1+x_2) + 2i\omega\Omega(y_1+y_2) = 0$$

$$-2i\omega\Omega(x_1+x_2) - \omega^2(y_1+y_2) = 0$$

$$\omega^2\left(R\frac{d\Omega^2}{dR} - \omega^2\right) + 4\omega^2\Omega^2 = 0$$

$$\omega^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 = \varkappa^2$$