[illegible]

EP 32

kovariantní zápis Eulerovy rovnice:

$$\frac{\partial U^\mu}{\partial t} + U^\nu U^\mu{}_{;\nu} = -\frac{1}{\rho} p_{,}{}^{\mu} - \Phi_{,}{}^{\mu} + \frac{1}{\rho} \Pi^{\mu\nu}{}_{;\nu}$$

$$U^\mu = \frac{\partial x^\mu}{\partial t} = (\dot{R}, \dot{\varphi}, \dot{z}) = (\dot{R}, \Omega, \dot{z})$$

kovariantní derivace (v obecně křivočarých souř.):

$$A^\mu{}_{;\nu} \equiv A^\mu{}_{,\nu} + \Gamma^\mu{}_{\rho\nu} A^\rho$$

$$A_{\mu;\nu} \equiv A_{\mu,\nu} - \Gamma^\rho{}_{\mu\nu} A_\rho$$

$$T^{\mu\nu}{}_{;\alpha} \equiv T^{\mu\nu}{}_{,\alpha} + \Gamma^\mu{}_{\rho\alpha} T^{\rho\nu} + \Gamma^\nu{}_{\rho\alpha} T^{\mu\rho}$$

Christoffelovy symboly:

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(g_{\gamma\alpha,\beta} + g_{\gamma\beta,\alpha} - g_{\alpha\beta,\gamma}\right)$$

metrický tenzor:

$$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta \qquad \tilde{g}_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial \tilde{x}^\alpha} \frac{\partial x^\nu}{\partial \tilde{x}^\beta}$$

cylindrické souřadnice:

$$g_{\mu\nu} = \begin{array}{cc} \begin{pmatrix} 1 & & 0 \\ & R^2 & \\ 0 & & 1 \end{pmatrix} & \begin{array}{c} R \\ \varphi \\ z \end{array} \end{array} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & 1/R^2 & \\ 0 & & 1 \end{pmatrix}$$

jediné nenulové složky afinní konexe:

$$\Gamma_{R\varphi\varphi} = -R \quad , \quad \Gamma_{\varphi R\varphi} = \Gamma_{\varphi\varphi R} = R$$

φ-tá složka Eulera:

$$U^\nu U^\varphi{}_{;\nu} = U^\varphi U^\varphi{}_{,\varphi} + U^\varphi g^{\varphi\varphi} \Gamma_{\varphi R\varphi} U^R + U^R U^\varphi{}_{,R} + U^R g^{\varphi\varphi} \Gamma_{\varphi\varphi R} U^\varphi +$$

$$+ U^z U^\varphi{}_{,z} + \cancel{U^z g^{\varphi\varphi} \Gamma_{\varphi\alpha z} U^\alpha}$$

$$= \cancel{U^\varphi U^\varphi{}_{,\varphi}} + U^R U^\varphi{}_{,R} + \frac{2}{R} U^\varphi U^R + \cancel{U^z U^\varphi{}_{,z}} =$$

$$= U^\varphi U^\varphi{}_{,\varphi} + U^R \frac{1}{R^2} \frac{\partial}{\partial R}\left(R^2 U^\varphi\right) + U^z U^\varphi{}_{,z}$$

$$\Pi^{\varphi\nu}{}_{;\nu} = \Pi^{\varphi\nu}{}_{,\nu} + g^{\varphi\varphi}\Gamma_{\varphi\rho\nu}\Pi^{\rho\nu} + g^{\mu\nu}\Gamma_{\mu\rho\nu}\Pi^{\varphi\rho}$$

$$= \Pi^{\varphi r}{}_{,r} + \Pi^{\varphi\varphi}{}_{,\varphi} + \Pi^{\varphi z}{}_{,z} + 2\frac{1}{r^2} r \Pi^{r\varphi} + \frac{1}{r^2} r \Pi^{r\varphi} =$$

$$= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3 \Pi^{\varphi r}\right) + \cancel{\Pi^{\varphi\varphi}{}_{,\varphi}} + \cancel{\Pi^{\varphi z}{}_{,z}}$$

Navier-Stokes

$$\Pi^{ik} = \eta\left(\frac{\partial u^i}{\partial x^k} + \frac{\partial u^k}{\partial x^i} - \frac{2}{3}\nabla\cdot u\,\delta^{ik}\right)$$

kovariantně:

$$\tilde\Pi^{\mu\nu} = \eta\left(U^{\mu;\nu} + U^{\nu;\mu} - \frac{2}{3} U^{\alpha}{}_{;\alpha}\, g^{\mu\nu}\right)$$

$\eta = \rho\nu$ = dynamická viskozita

(odvodit například z transformačních pravidel,
$\tilde A^{\mu\nu} = \frac{\partial \tilde x^\mu}{\partial x^\rho}\frac{\partial \tilde x^\nu}{\partial x^\varkappa} A^{\rho\varkappa}$)

Zajímá nás $\Pi^{\varphi r}$:

$$\frac{1}{\eta}\Pi^{\varphi r} = g^{rr}U^{\varphi}{}_{,r} + g^{rr}\Gamma^{\varphi}{}_{r\nu}U^{\nu} + g^{\varphi\varphi}U^{r}{}_{,\varphi} + g^{\varphi\varphi}\Gamma^{r}{}_{\varphi\nu}U^{\nu} - \frac{2}{3}U^{\alpha}{}_{,\alpha}\,\underbrace{g^{\varphi r}}_{=0}$$

$$= U^{\varphi}{}_{,r} + g^{\varphi\varphi}\Gamma_{\varphi r\varphi}U^{\varphi} + \frac{1}{r^2}\left(U^{r}{}_{,\varphi} + g^{rr}\Gamma_{r\varphi\varphi}U^{\varphi}\right) =$$

$$= U^{\varphi}{}_{,r} + \cancel{\frac{1}{r}U^{\varphi}} + \frac{1}{r^2}U^{r}{}_{,\varphi} - \cancel{\frac{1}{r}U^{\varphi}}$$

$$\Pi^{\varphi r} = \eta\frac{\partial\Omega}{\partial r}$$

dosazení do Eulera

$$\frac{\partial U^\varphi}{\partial t} + U^r\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 U^\varphi\right) = \frac{1}{\rho}\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\eta\frac{\partial\Omega}{\partial r}\right)$$

rovnice kontinuity:

$$\frac{\partial\rho}{\partial t} + \left(\rho U^\mu\right)_{;\mu} = 0 \quad\to\quad \frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\rho U^r\right) = 0 \quad /\cdot rU^\varphi$$

$$\to r\rho\frac{\partial U^\varphi}{\partial t} + rU^\varphi\frac{\partial\rho}{\partial t} + \frac{1}{r^2}\left(r\rho U^r\right)\frac{\partial}{\partial r}\left(r^2\Omega\right) + \frac{1}{r^2}\left(r^2\Omega\right)\frac{\partial}{\partial r}\left(r\rho U^r\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^3\eta\frac{\partial\Omega}{\partial r}\right)$$

$$\to r^3\left(\rho\frac{\partial\Omega}{\partial t} + \Omega\frac{\partial\rho}{\partial t}\right) + \frac{\partial}{\partial r}\left(r^3\rho U^r\Omega\right) = \frac{\partial}{\partial r}\left(r^3\rho\nu\frac{\partial\Omega}{\partial r}\right)$$

srovnej s Eulerem na EP26 ⊕ $v_\varphi = r\Omega$